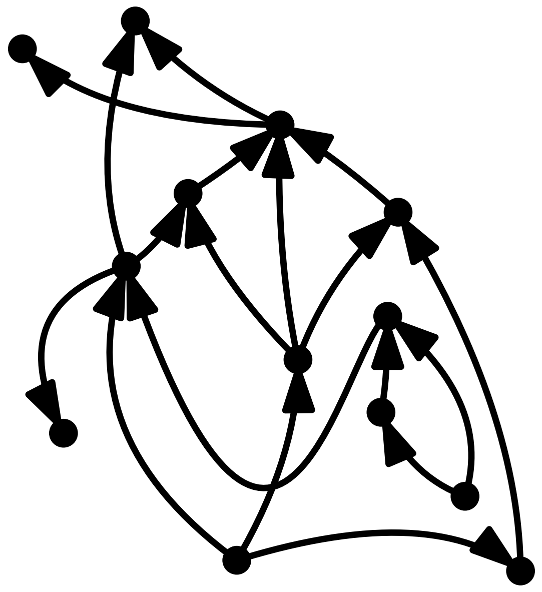
# Upward planar drawings with three or more slopes

Jonathan Klawitter · Johannes Zink

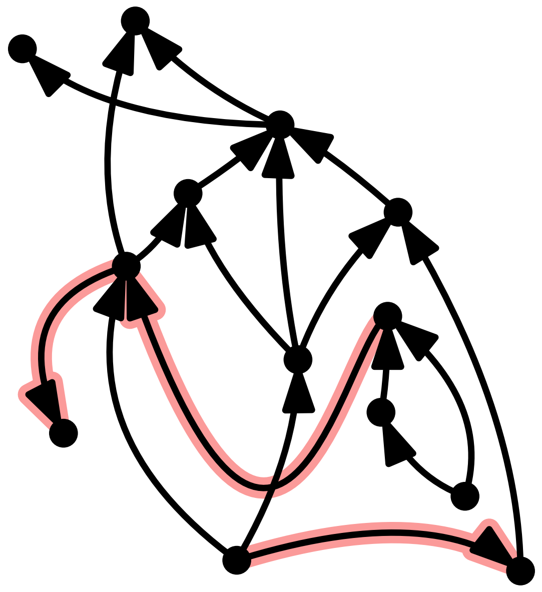
# Upward planar $k$-slope drawings

$G$



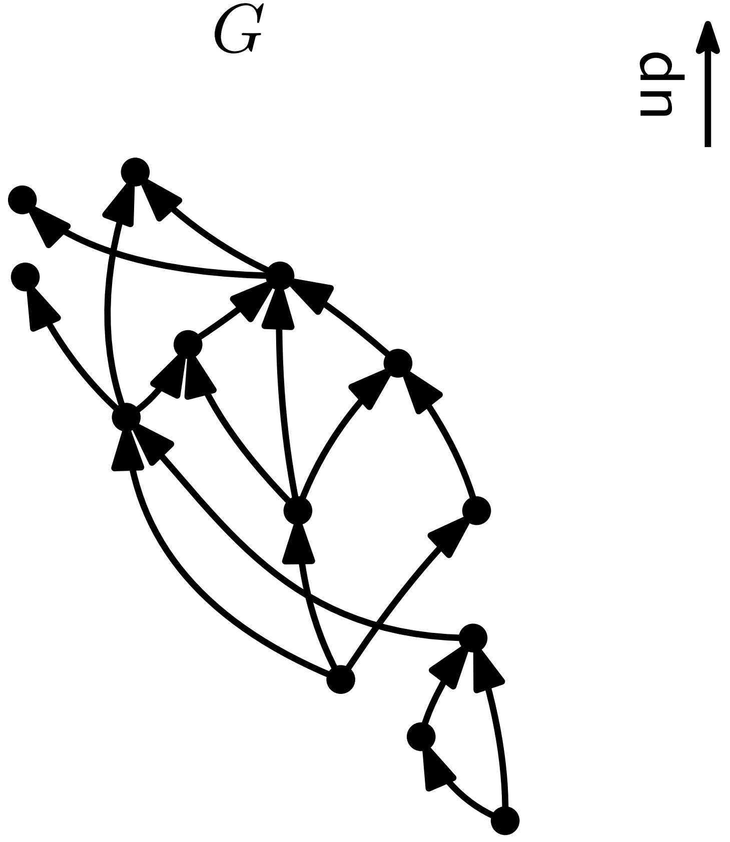
# Upward planar $k$-slope drawings

$G$

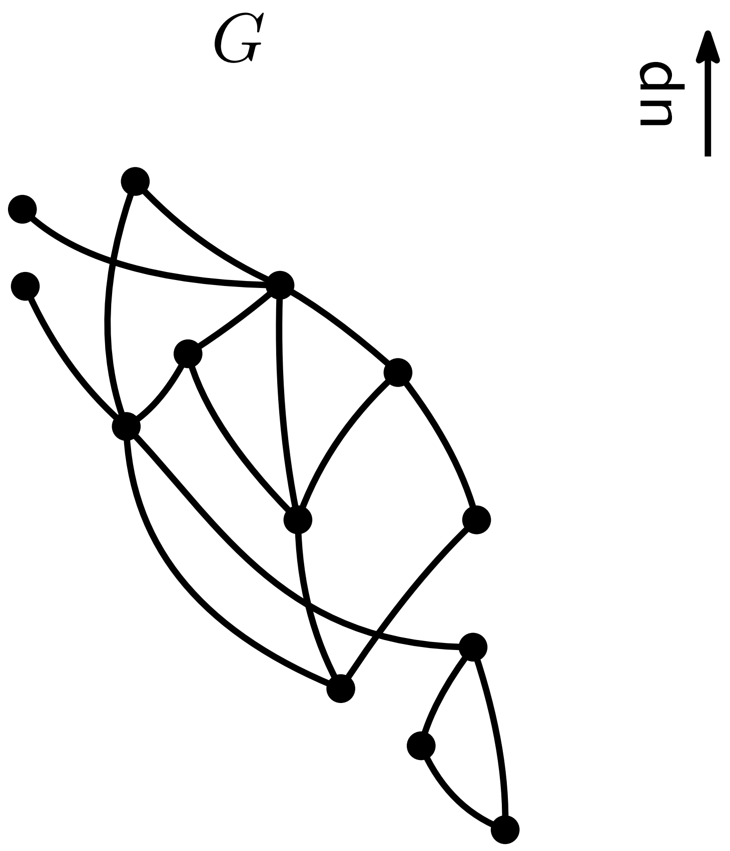


# Upward planar $k$-slope drawings

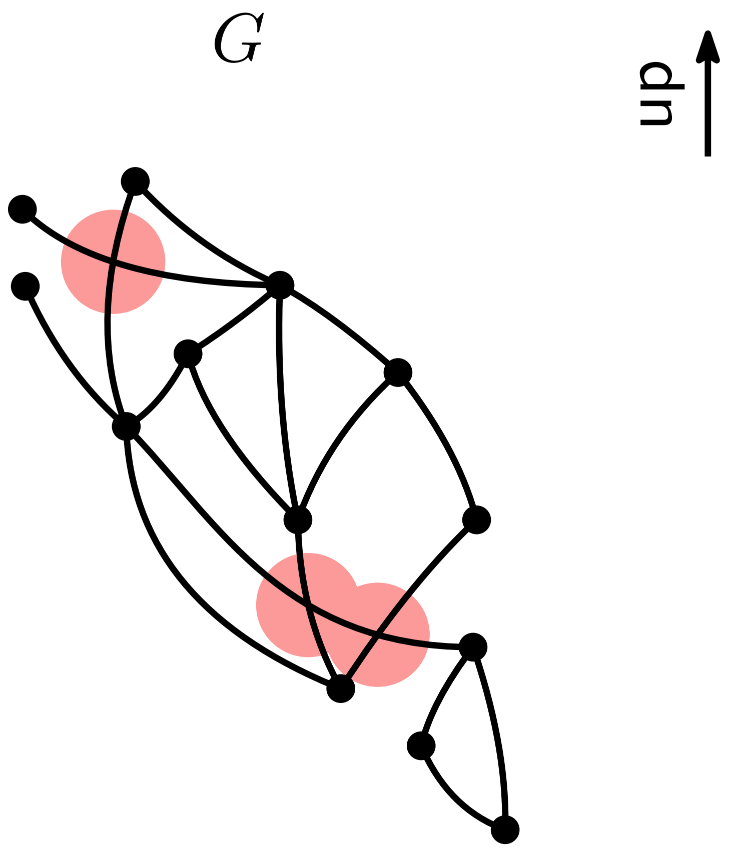
$G$



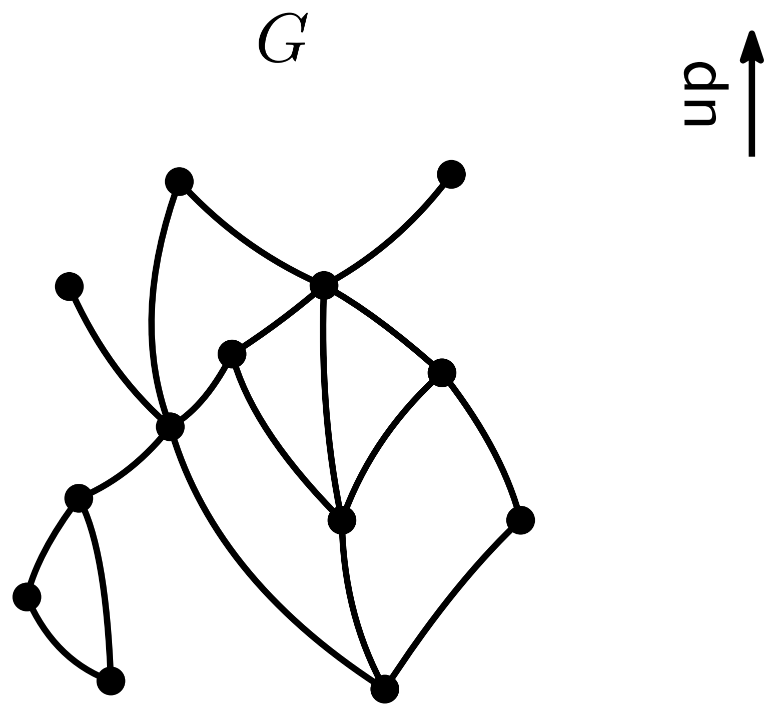
# Upward planar $k$-slope drawings



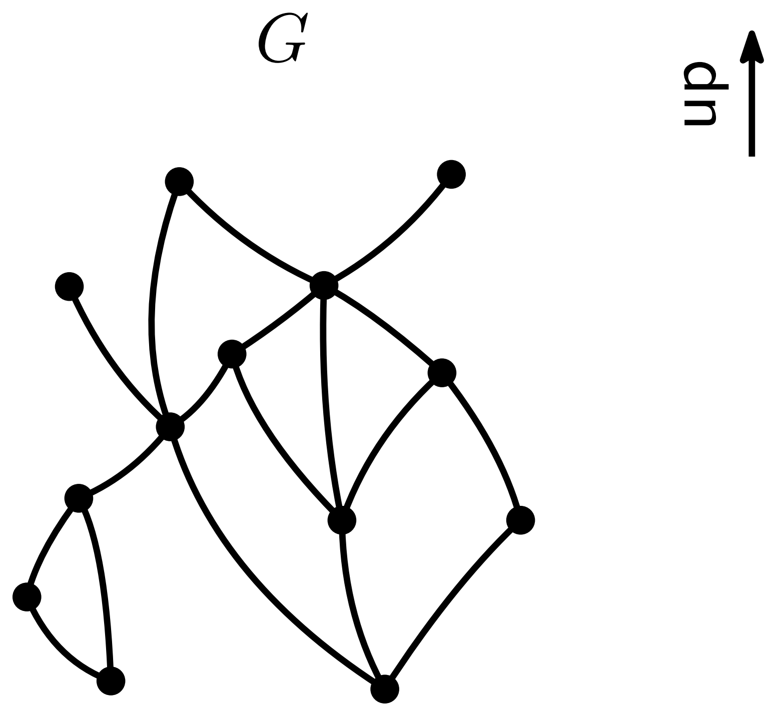
# Upward planar $k$-slope drawings

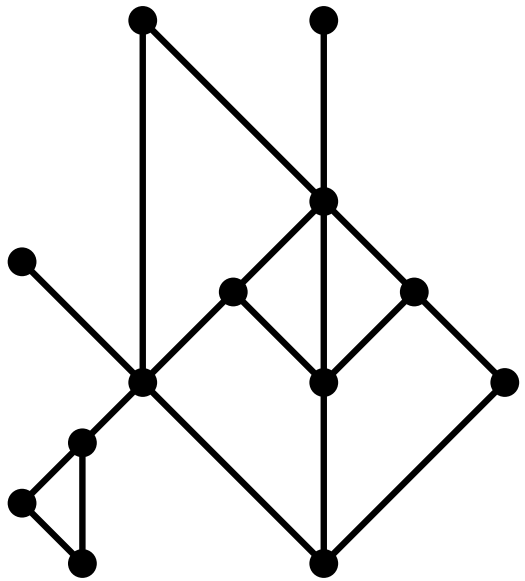


# Upward planar $k$-slope drawings



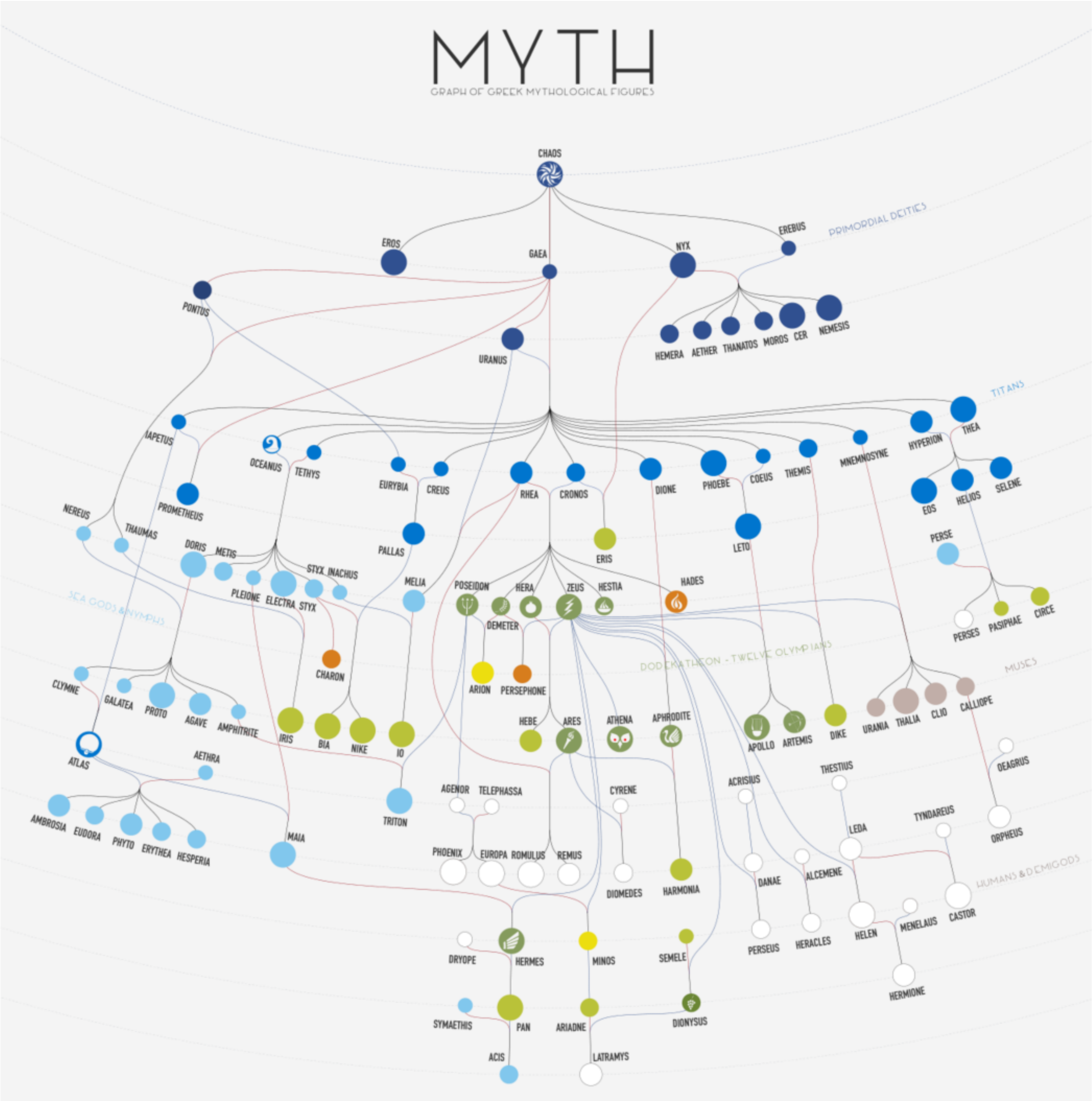
# Upward planar $k$-slope drawings

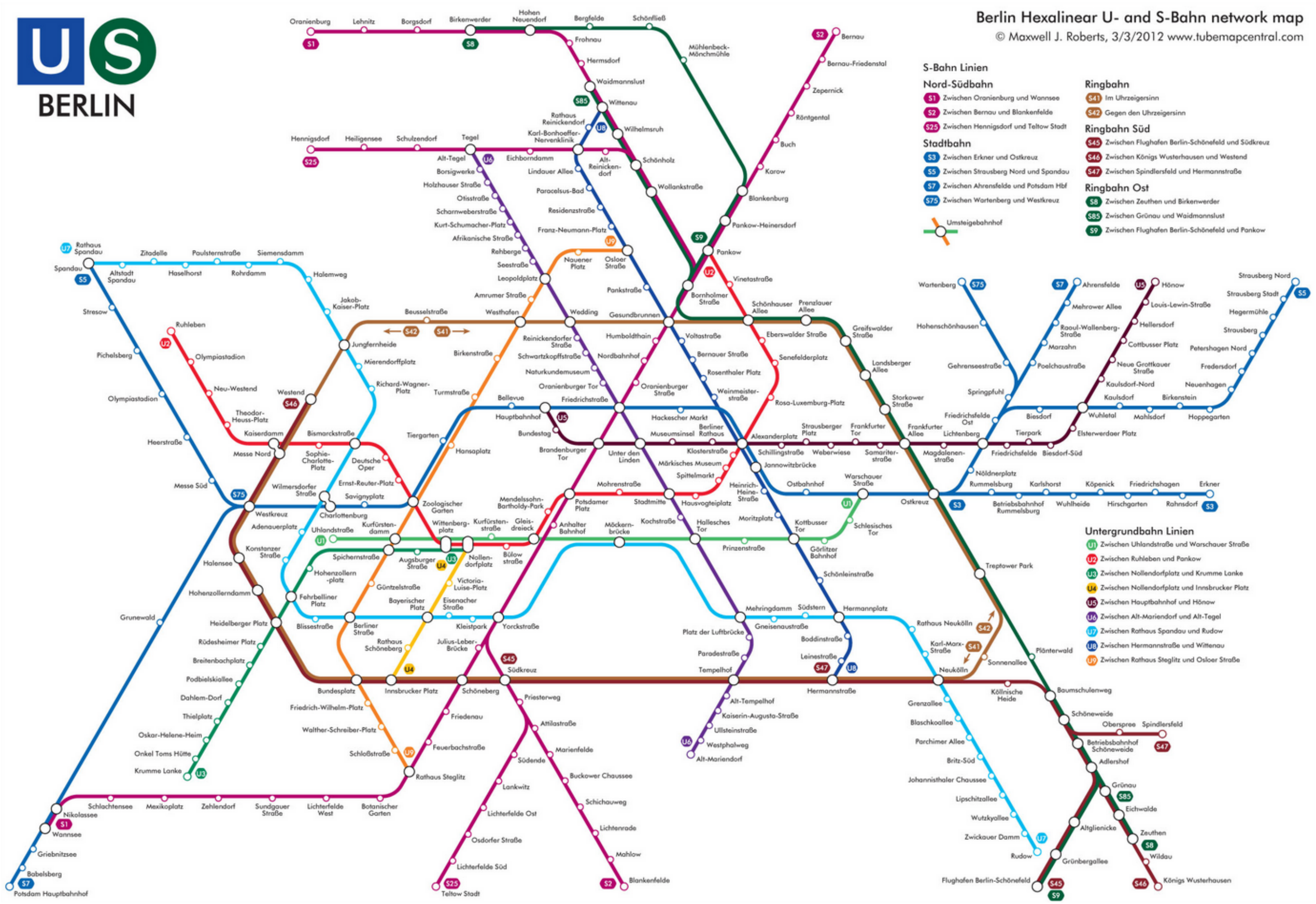


$k = 3$

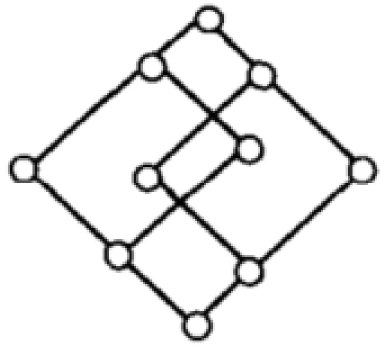
# Motivation


MYTH
GRAPH OF GREEK MYTHOLOGICAL FIGURES
CHAOS
EROS
GAEA
NYX
EREBUS
PRIMORDIAL DEITIES
URANUS
TITANS
ZEUS
HERA
POSEIDON
HADES


U S
BERLIN
Berlin Hexalinear U- and S-Bahn network map
© Maxwell J. Roberts, 3/3/2012 www.tubemapcentral.com
S-Bahn Linien
Untergrundbahn Linien

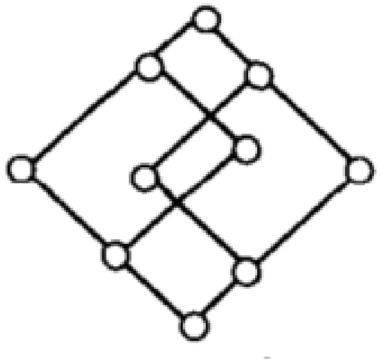
# Related work

- Czyzowicz et al. '90:
  Every finite planar lattice with maximum up/down-degree 2, has an upward planar 2-slope drawing.
- Czyzowicz '91:
  Maximum up/downdegree of lattice is only a lower bound for slope number.

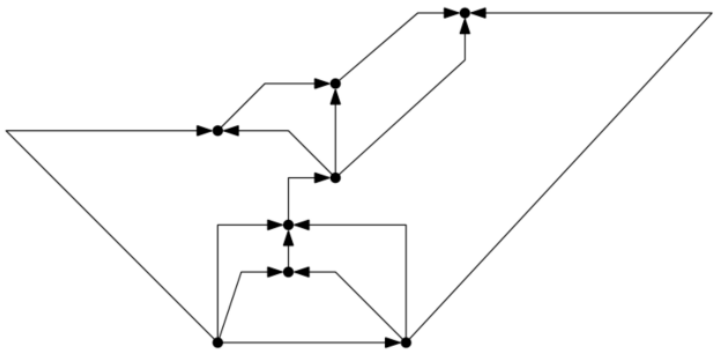


# Related work

- Czyzowicz et al. '90:
  Every finite planar lattice with maximum up/down-degree 2, has an upward planar 2-slope drawing.



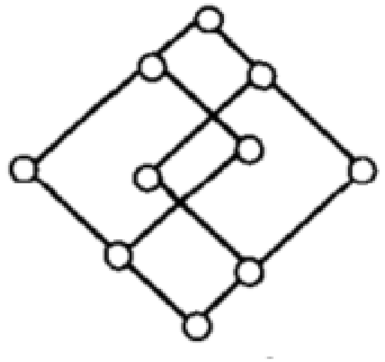
- Czyzowicz '91:
  Maximum up/downdegree of lattice is only a lower bound for slope number.
- Bekos et al. '18:
  Every bitonic st-graph with maximum in-/outdegree $k$ admits an upward planar 1-bend $k$-slope drawing.



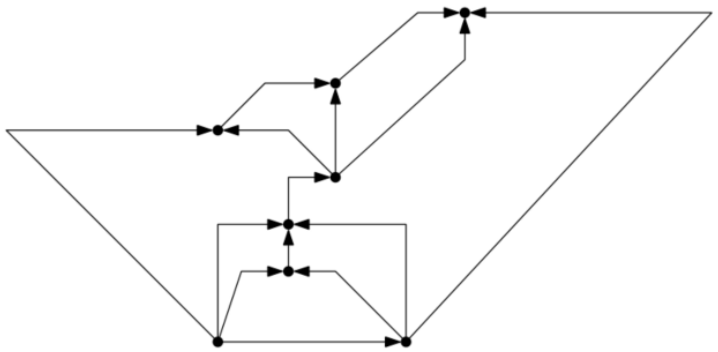
- Di Giacomo et al. '20 ('16):
  Every series-parallel digraph with maximum in-/outdegree $k$ admits an upward planar 1-bend $k$-slope drawing.

# Related work

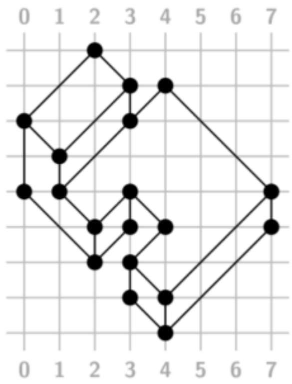
- Czyzowicz et al. '90:
  Every finite planar lattice with maximum up/down-degree 2, has an upward planar 2-slope drawing.



- Czyzowicz '91:
  Maximum up/downdegree of lattice is only a lower bound for slope number.
- Bekos et al. '18:
  Every bitonic st-graph with maximum in-/outdegree $k$ admits an upward planar 1-bend $k$-slope drawing.



- Di Giacomo et al. '20 ('16):
  Every series-parallel digraph with maximum in-/outdegree $k$ admits an upward planar 1-bend $k$-slope drawing.




- Brückner et al. '19:
  Considered level-planar drawings with a fixed slope set.

# Problem

Let $G$ be a digraph given with/without upward planar embedding.

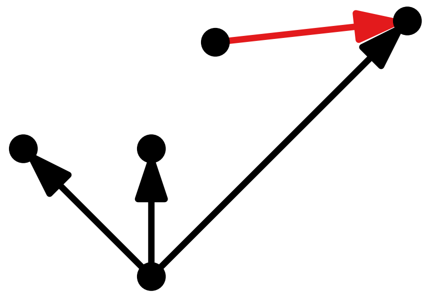
- Does $G$ admit an upward planar $k$-slope drawing?
- If so, can we construct one?

# Problem

Let $G$ be a digraph given with/without upward planar embedding.

- Does $G$ admit an upward planar $k$-slope drawing?
- If so, can we construct one?

$k = 3$:



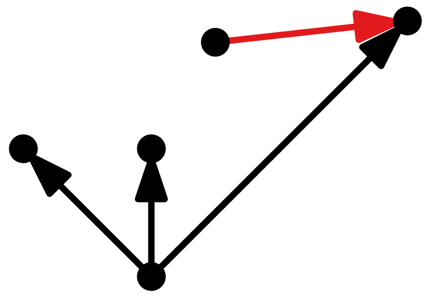
bad embedding

# Problem

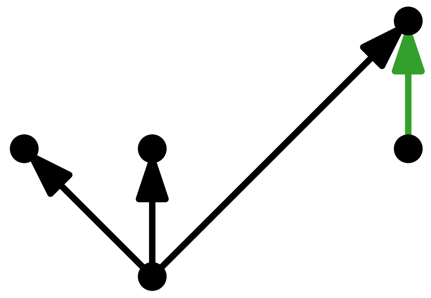
Let $G$ be a digraph given with/without upward planar embedding.

- Does $G$ admit an upward planar $k$-slope drawing?
- If so, can we construct one?

$k = 3$:



bad embedding



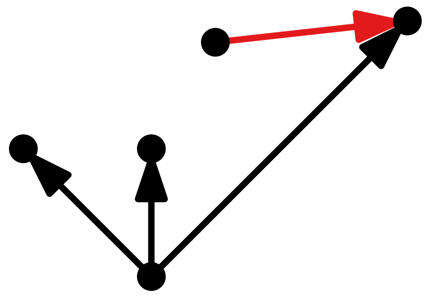
good embedding

# Problem

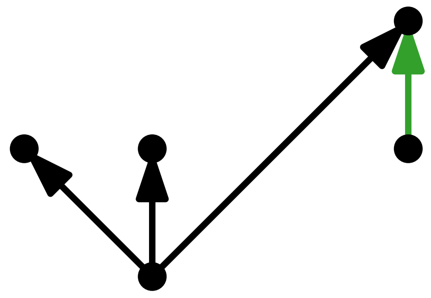
Let $G$ be a digraph given with/without upward planar embedding.

- Does $G$ admit an upward planar $k$-slope drawing?
- If so, can we construct one?

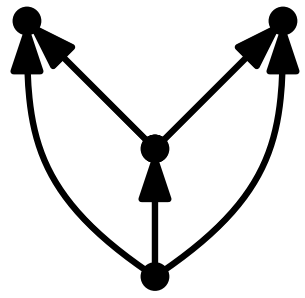
$k = 3$:



bad embedding



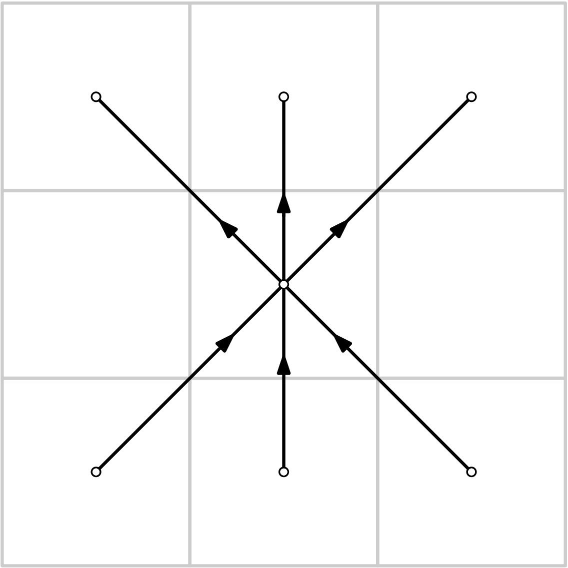
good embedding



no good embedding ex.

# Trees

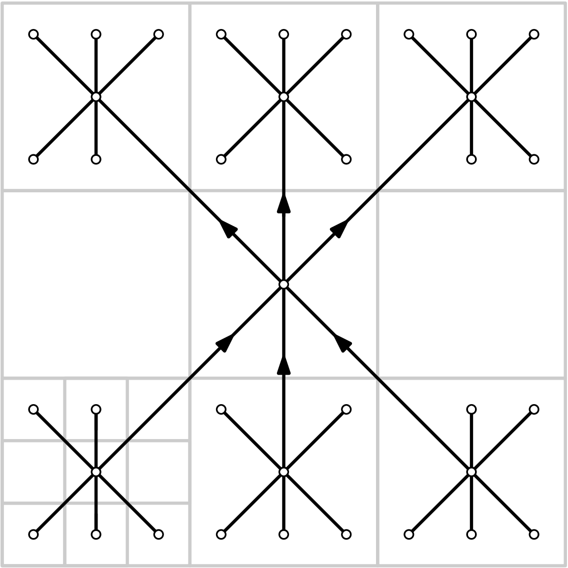
**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid



on the grid

# Trees

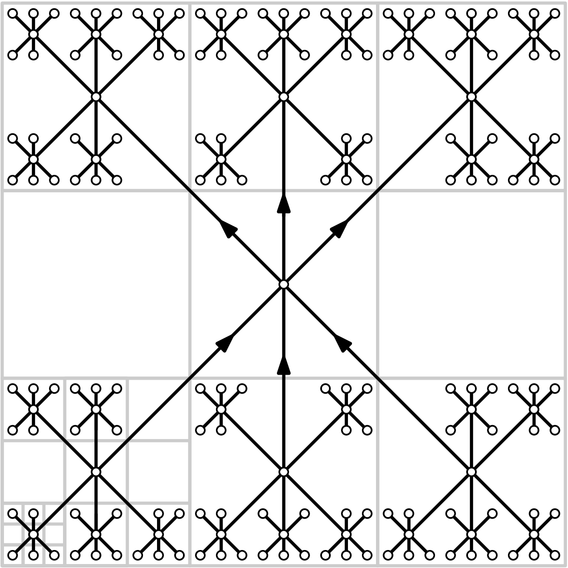
**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid



on the grid

# Trees

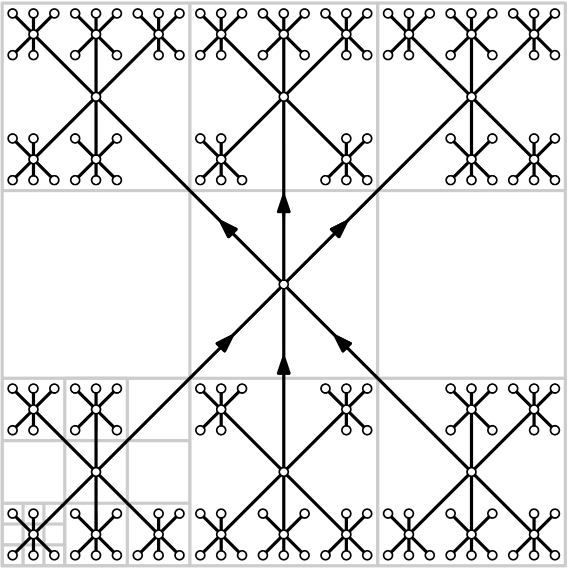
**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid

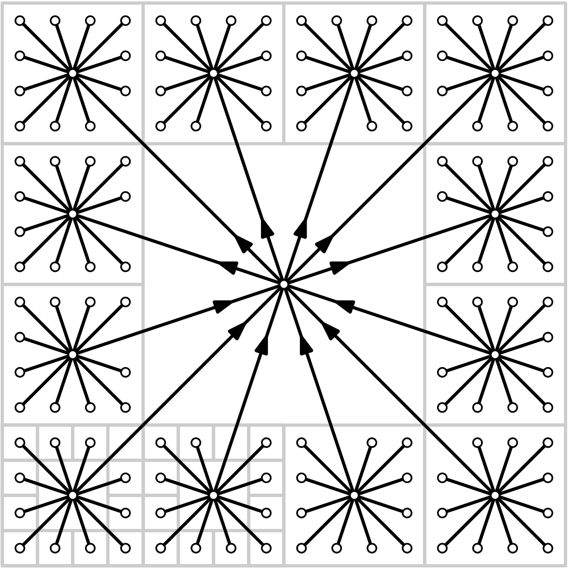


on the grid

# Trees

**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid

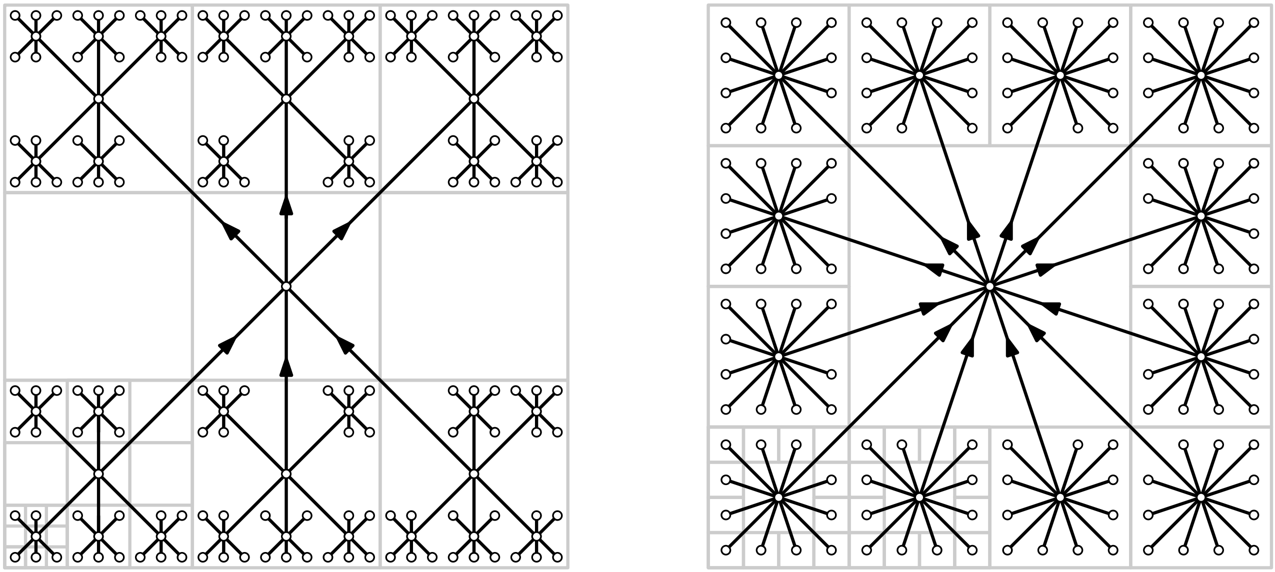




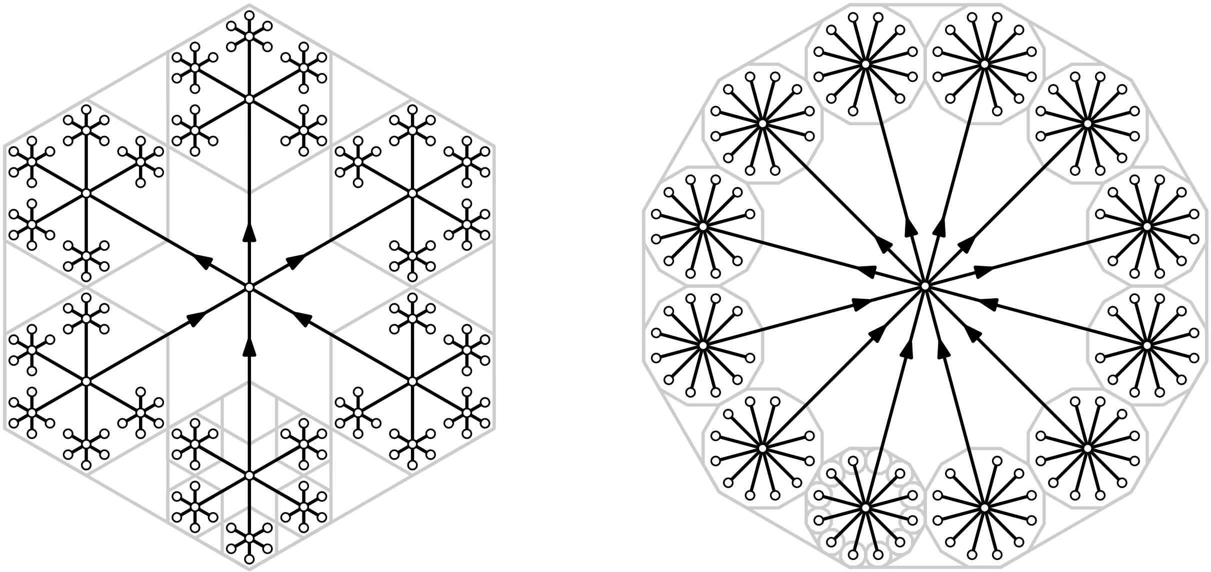
on the grid

# Trees

**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid or with uniform angles.



on the grid

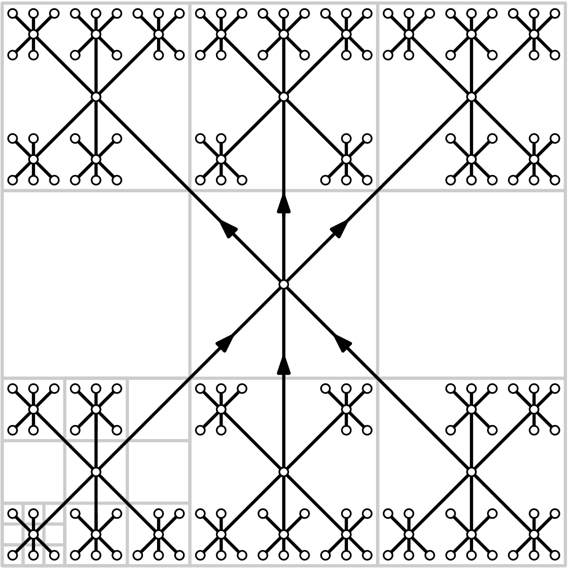
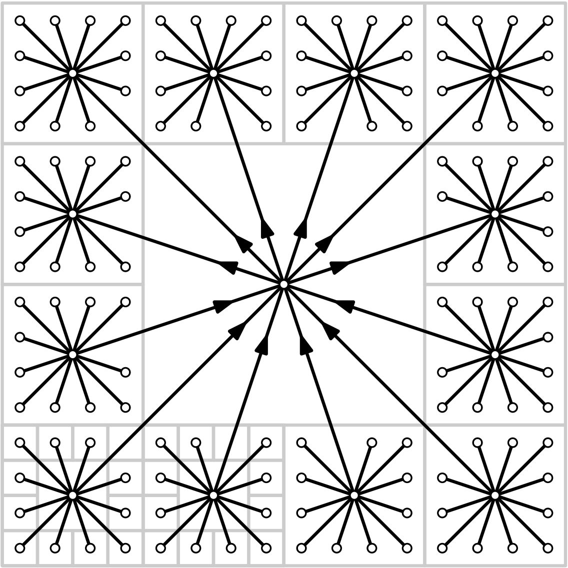


uniform angles

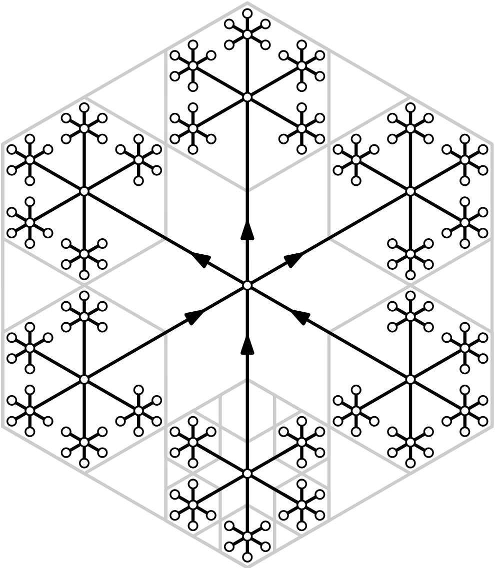
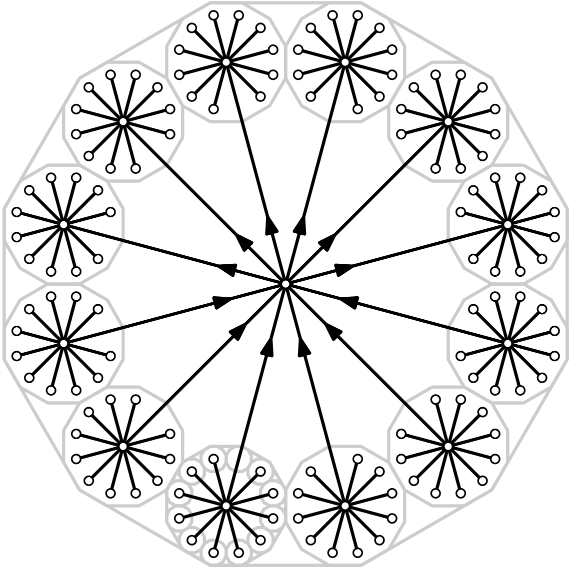
# Trees

**Theorem.**
Every unordered tree $T$ with max. in- and outdegree $k$ admits an upward planar $k$-slope drawing on the grid or with uniform angles.

**Theorem.**
The upward plane slope number of an ordered tree $T$ can be determined in linear time.

on the grid

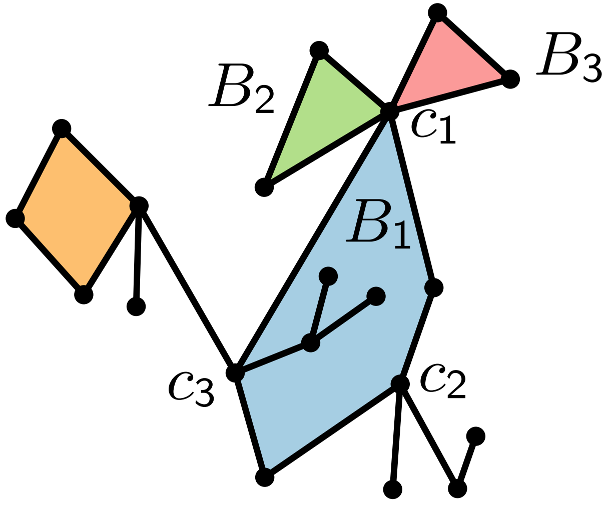



uniform angles

# Cactus digraphs

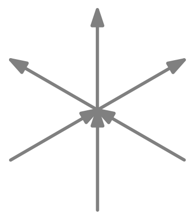
**Theorem.**

For an $n$-vertex upward planar (plane) cactus $G$,

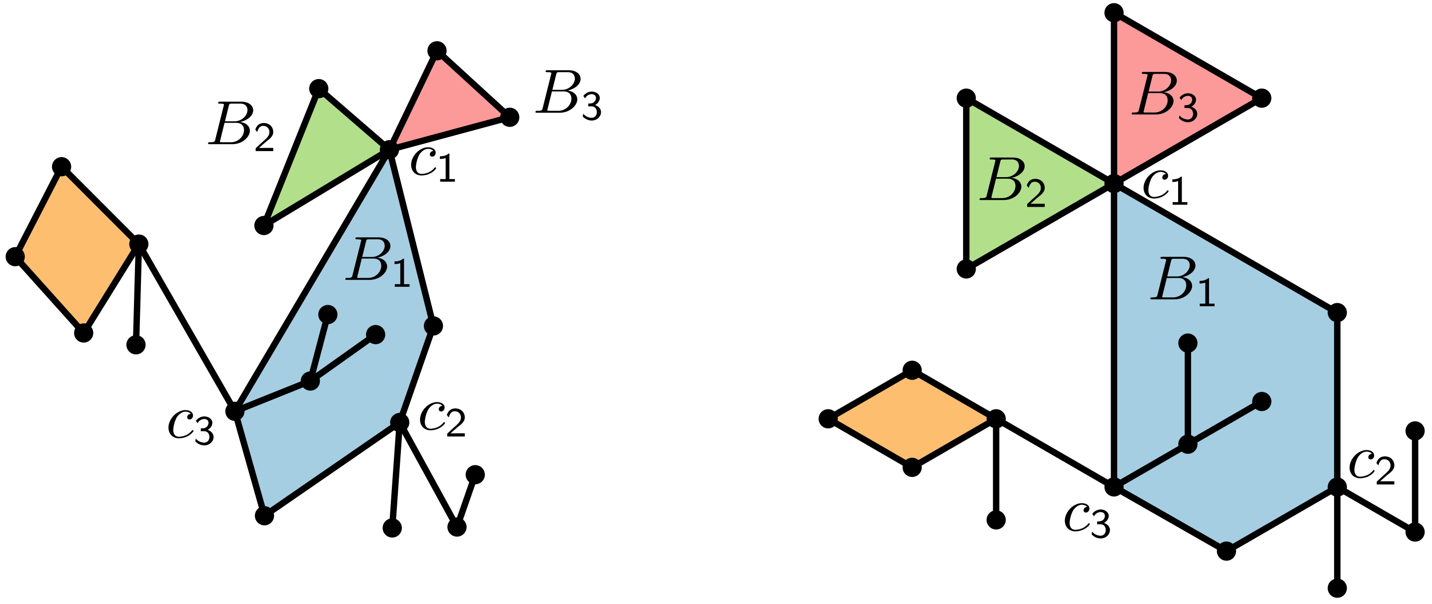


# Cactus digraphs

**Theorem.**
For an $n$-vertex upward planar (plane) cactus $G$,
it can be tested whether $G$ admits an upward planar
$k$-slope drawing with uniform angles in $\mathcal{O}(k^4 n^2)$ time.

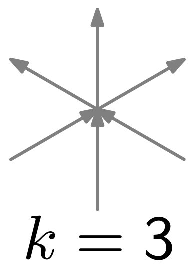


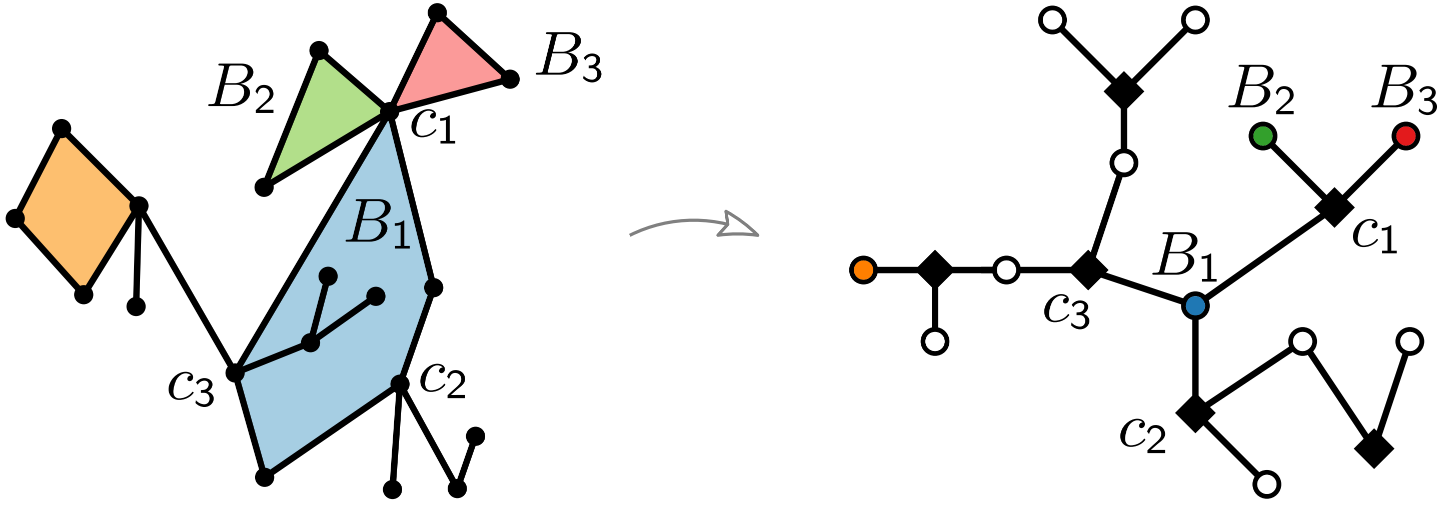
$k = 3$



# Cactus digraphs

**Theorem.**
For an $n$-vertex upward planar (plane) cactus $G$,
it can be tested whether $G$ admits an upward planar
$k$-slope drawing with uniform angles in $\mathcal{O}(k^4 n^2)$ time.

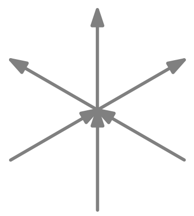


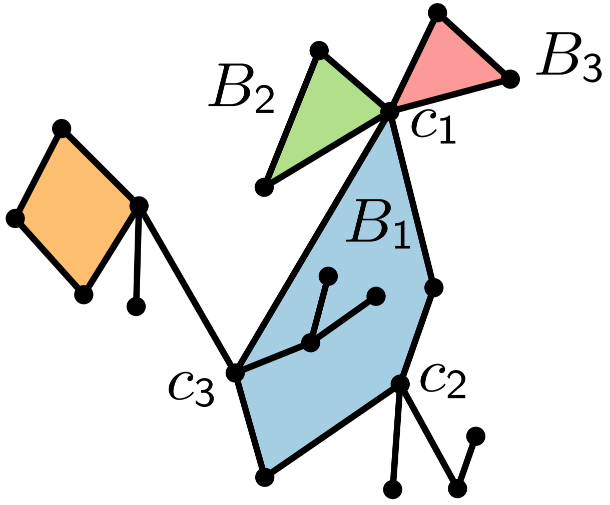



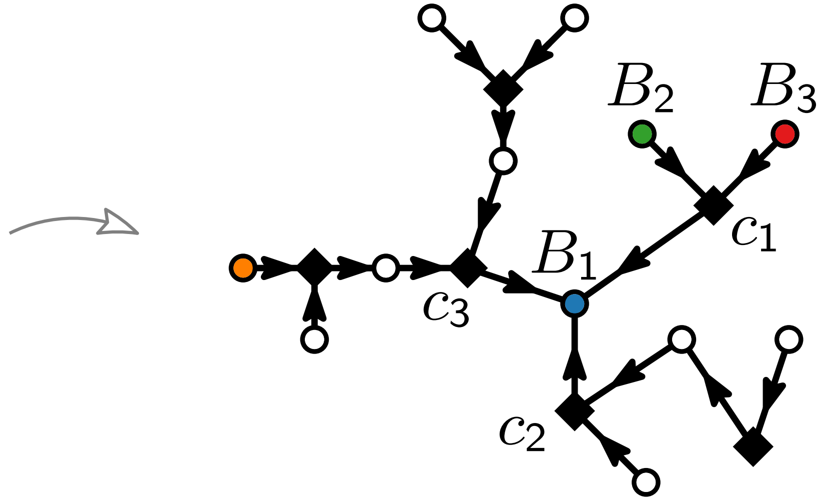

block-cut tree

# Cactus digraphs

**Theorem.**
For an $n$-vertex upward planar (plane) cactus $G$,
it can be tested whether $G$ admits an upward planar
$k$-slope drawing with uniform angles in $\mathcal{O}(k^4 n^2)$ time.



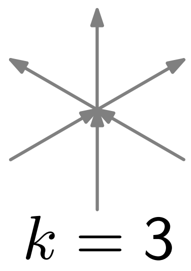



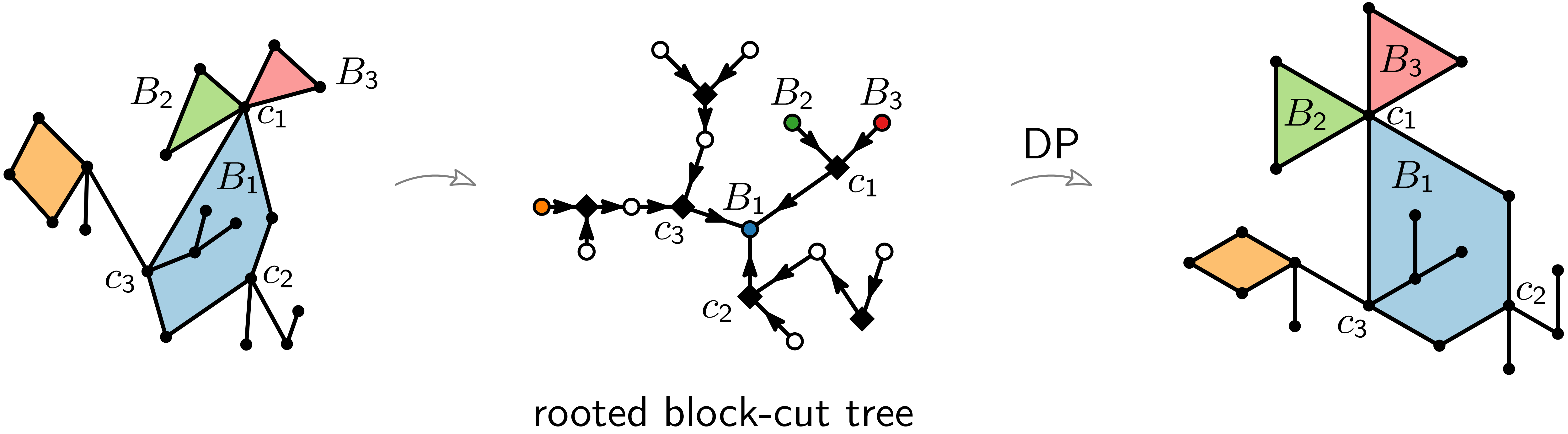




rooted block-cut tree

# Cactus digraphs

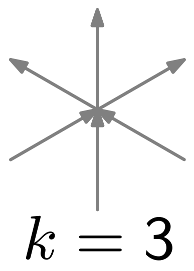
**Theorem.**
For an $n$-vertex upward planar (plane) cactus $G$, it can be tested whether $G$ admits an upward planar $k$-slope drawing with uniform angles in $\mathcal{O}(k^4 n^2)$ time.

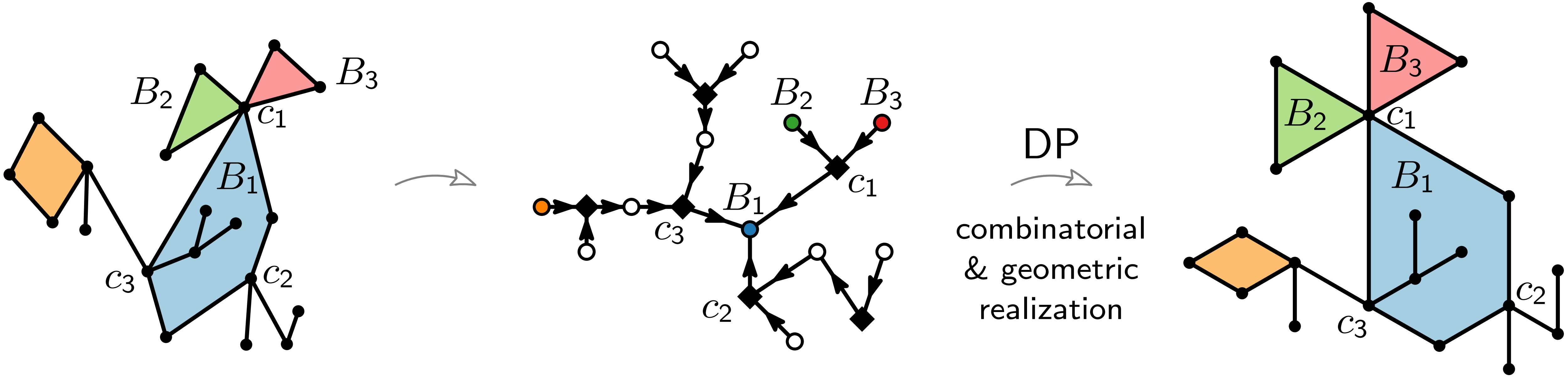


# Cactus digraphs

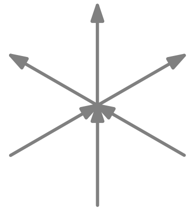
**Theorem.**
For an $n$-vertex upward planar (plane) cactus $G$,
it can be tested whether $G$ admits an upward planar
$k$-slope drawing with uniform angles in $\mathcal{O}(k^4 n^2)$ time.



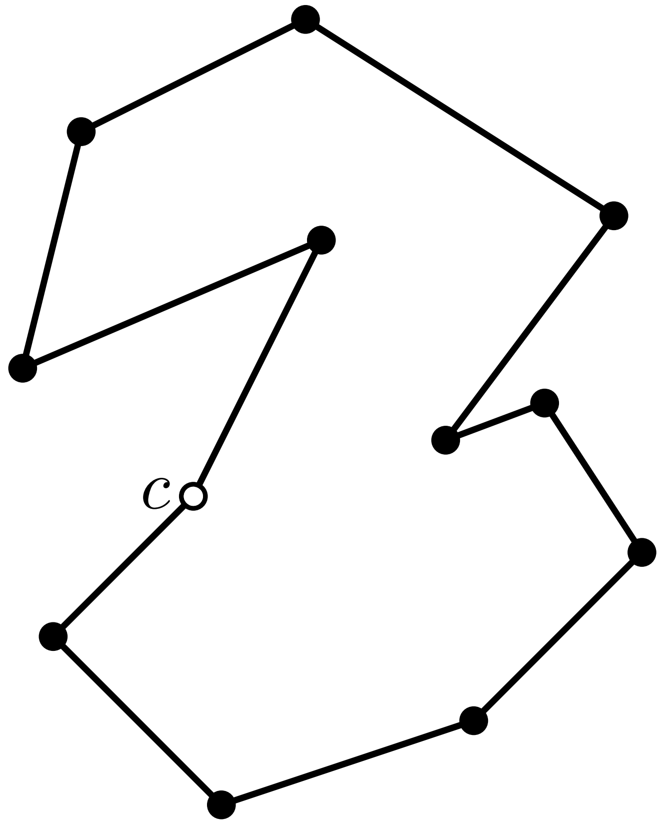




rooted block-cut tree

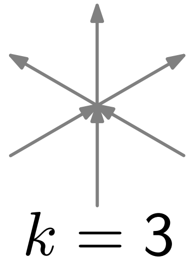
# Cactus digraphs – cycle block



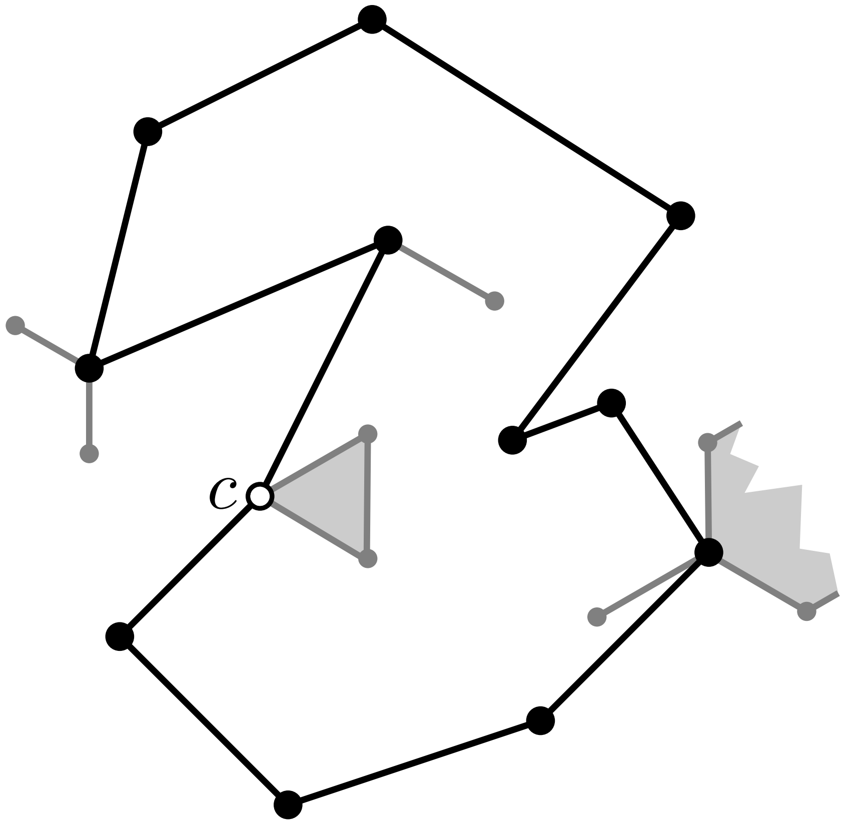

- Combinatorial realization



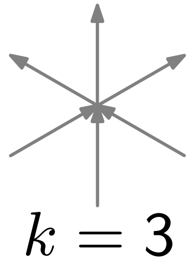
# Cactus digraphs – cycle block



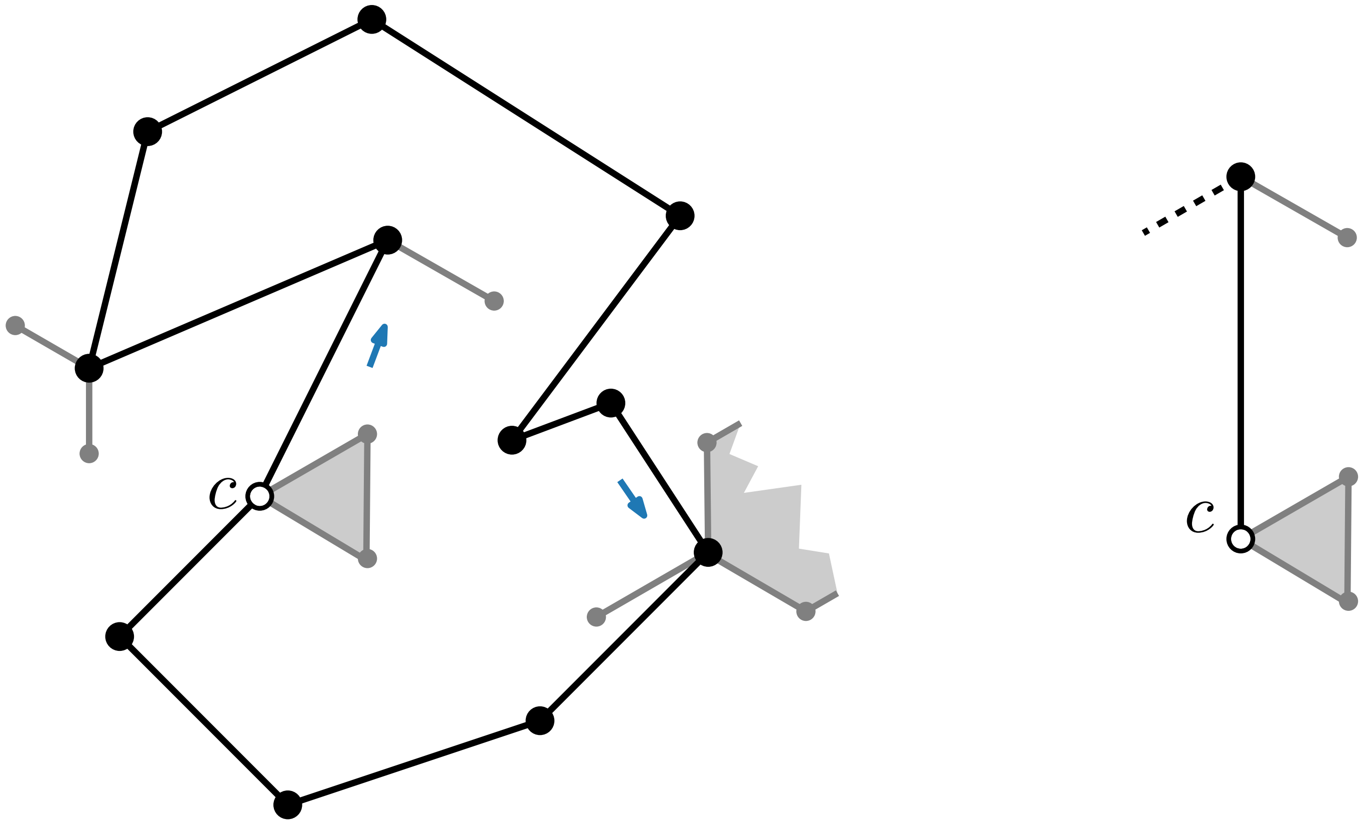

- Combinatorial realization



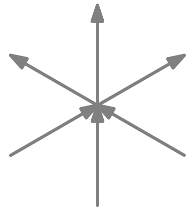
# Cactus digraphs – cycle block



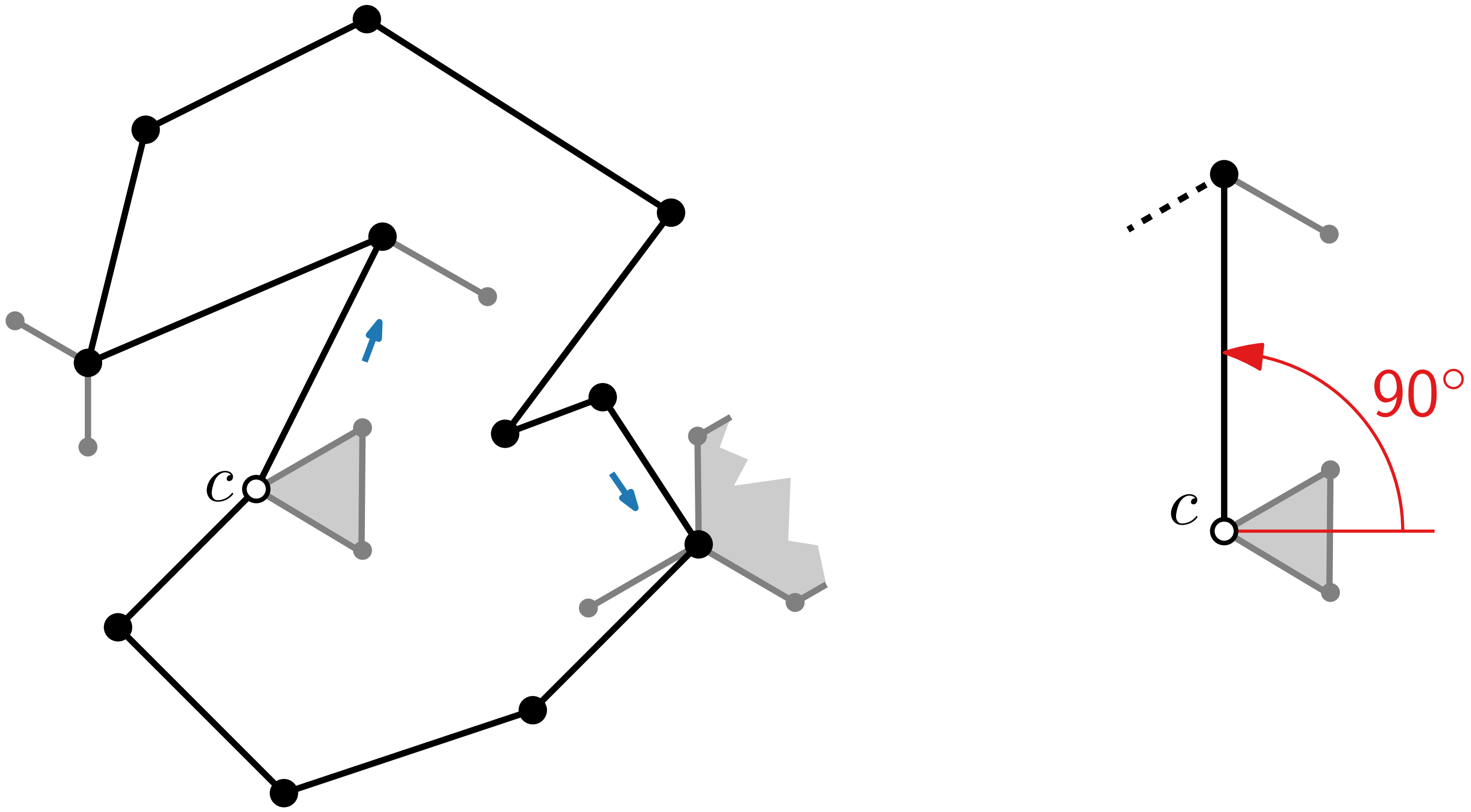

- Combinatorial realization



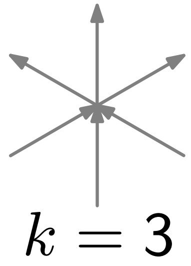
# Cactus digraphs – cycle block



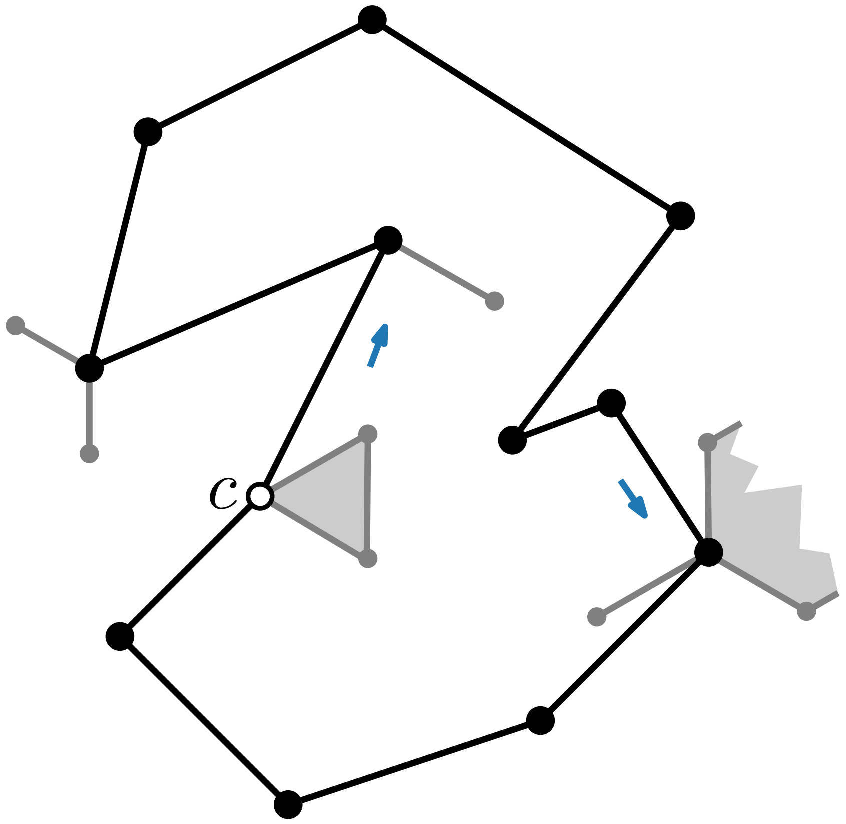

- Combinatorial realization

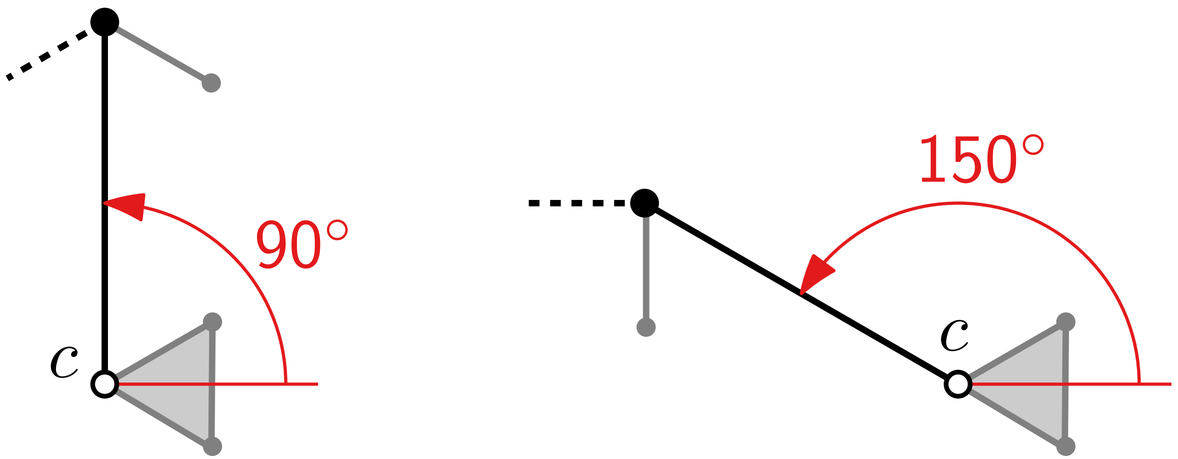


# Cactus digraphs – cycle block

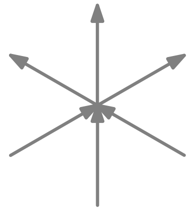



- Combinatorial realization

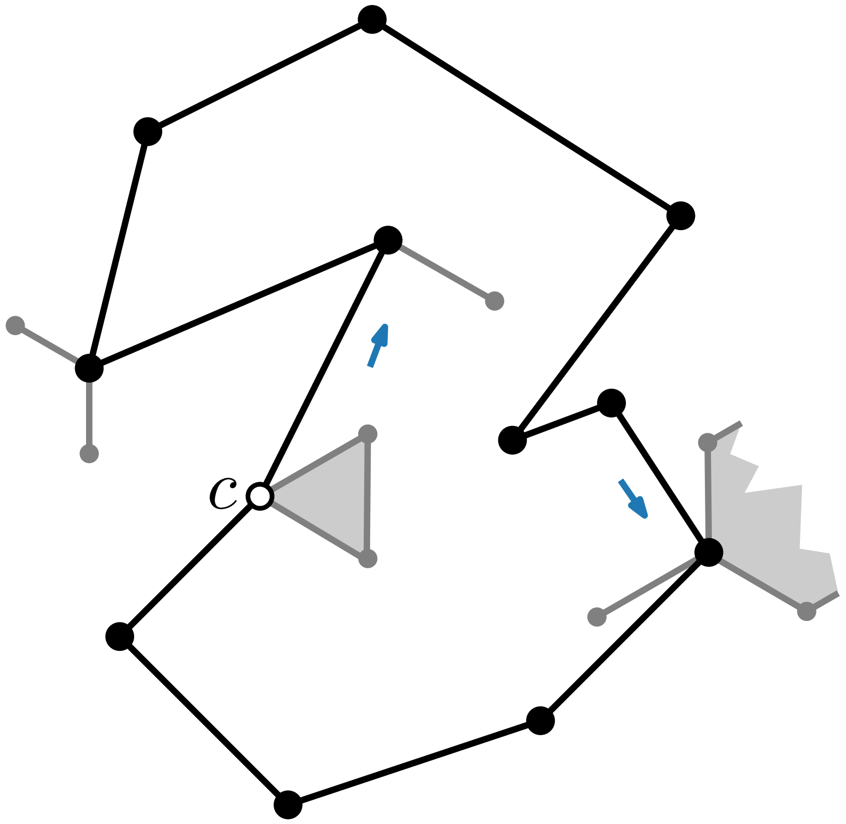


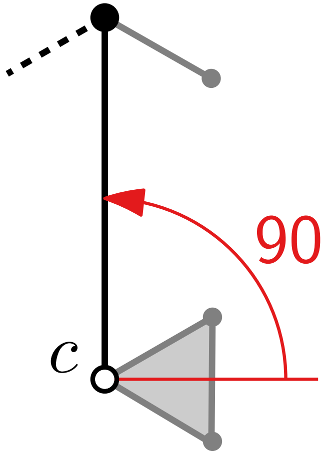



# Cactus digraphs – cycle block

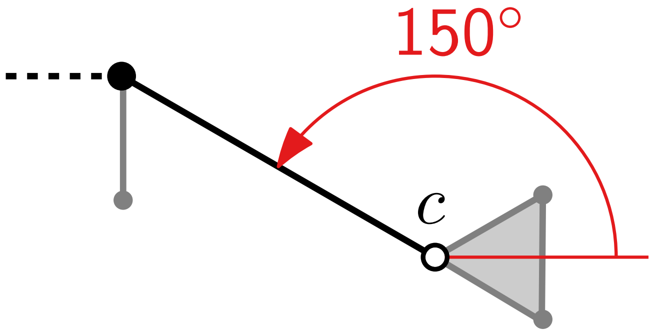



- Combinatorial realization



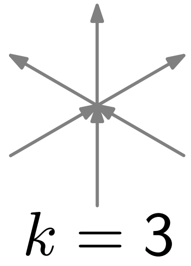




fixed embedding

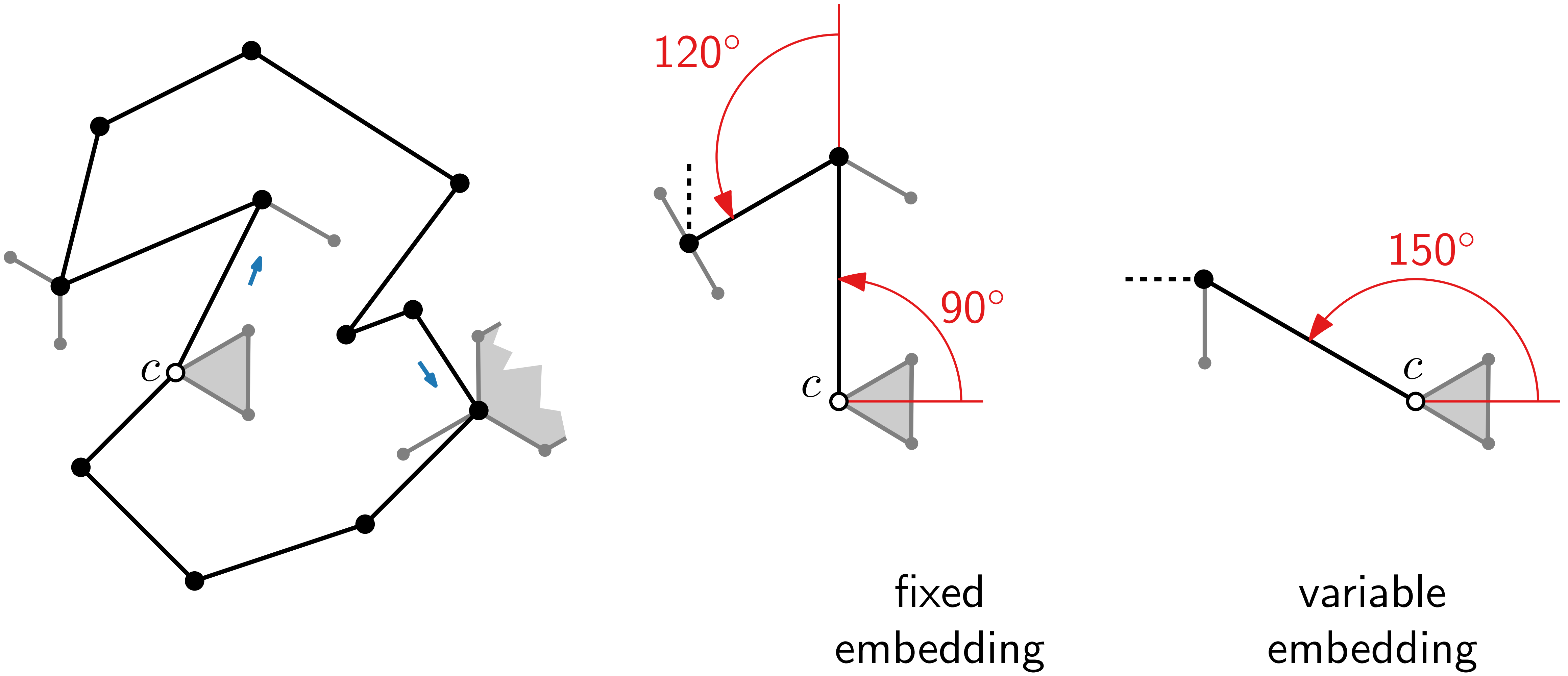



variable embedding

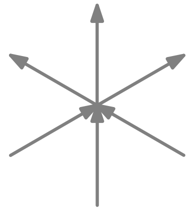
# Cactus digraphs – cycle block



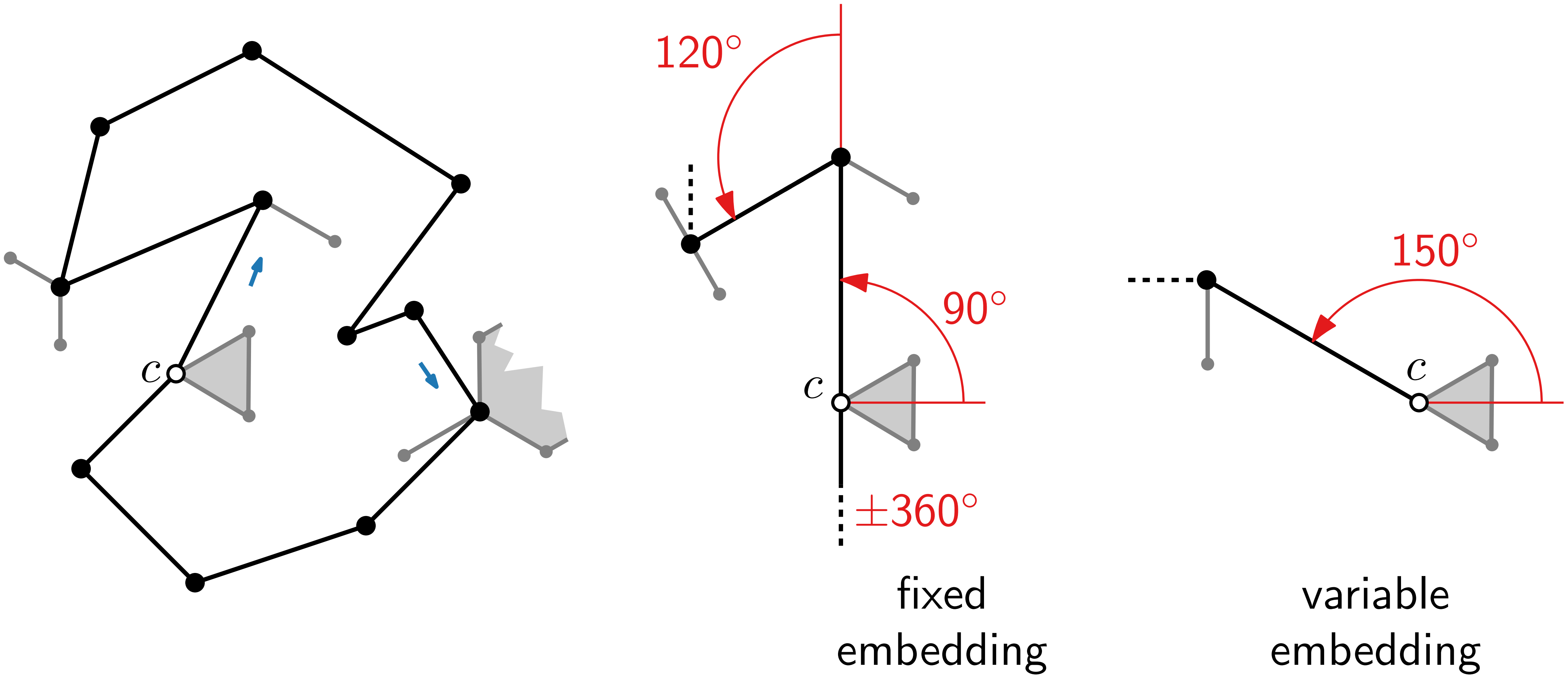

- Combinatorial realization



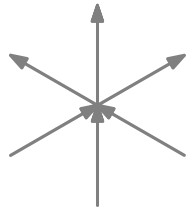
# Cactus digraphs – cycle block



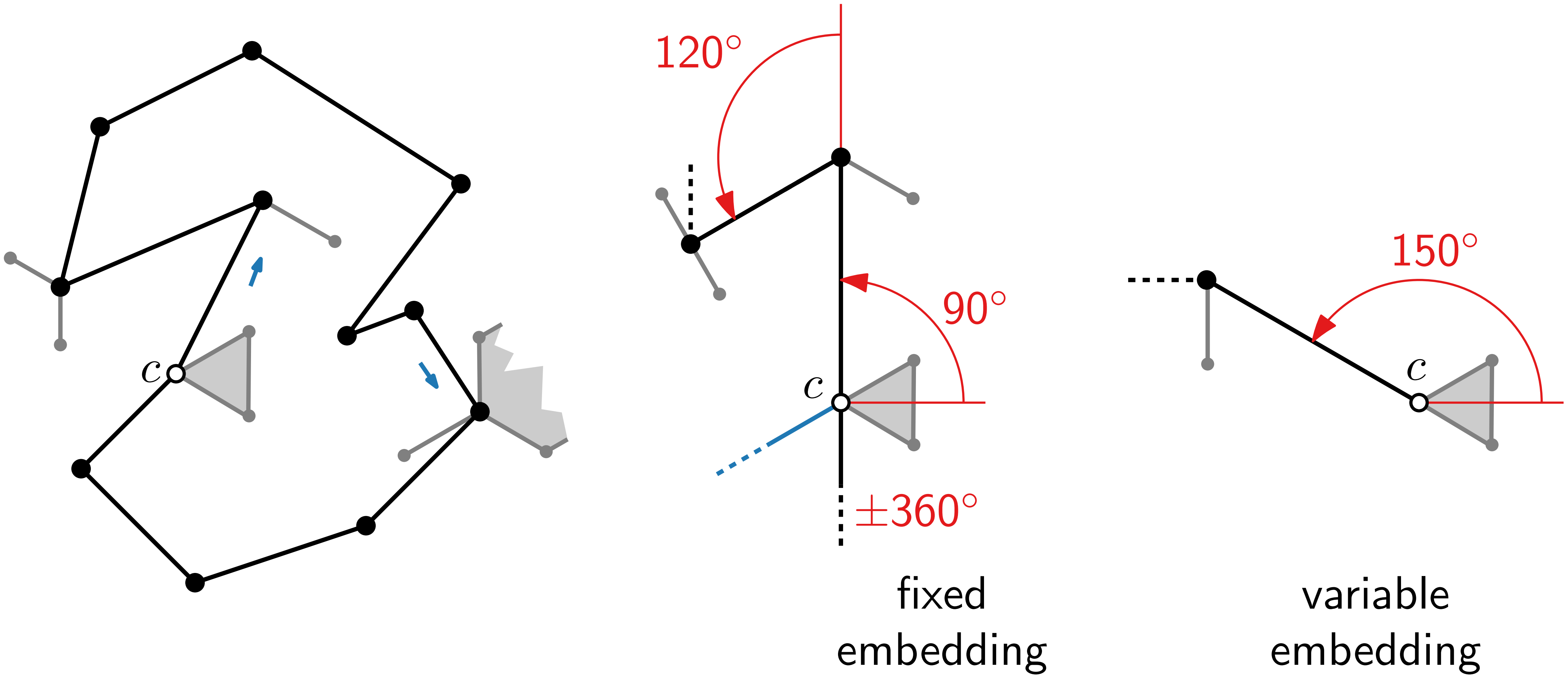

- Combinatorial realization



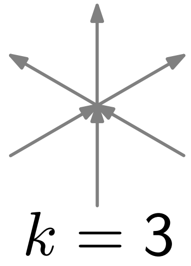
# Cactus digraphs – cycle block



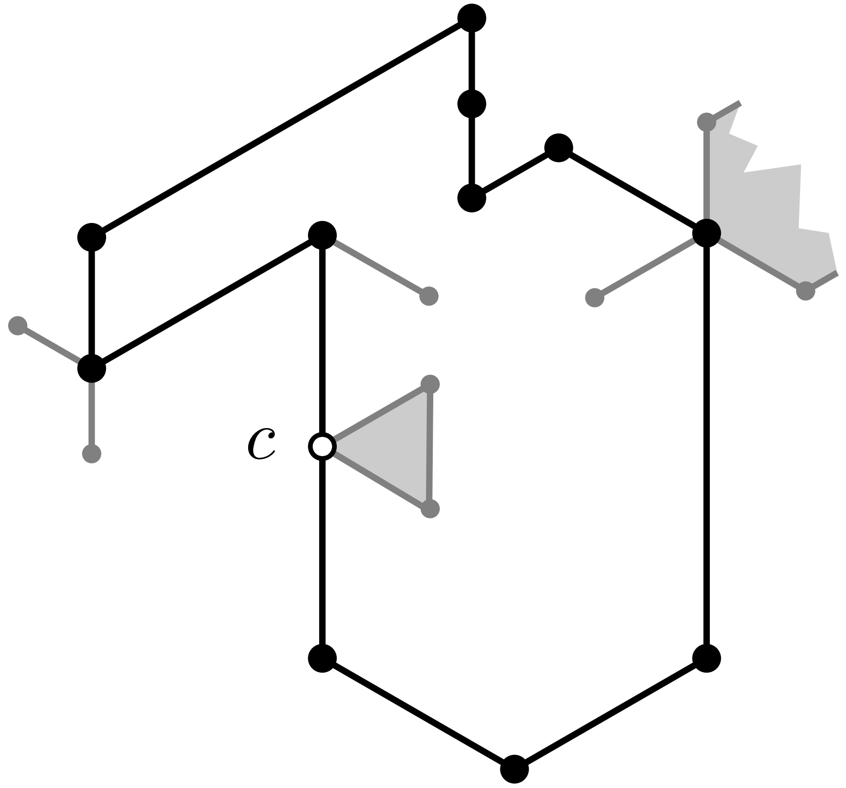

- Combinatorial realization



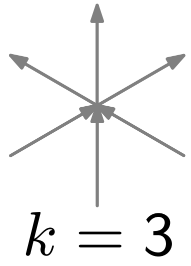
# Cactus digraphs – cycle block



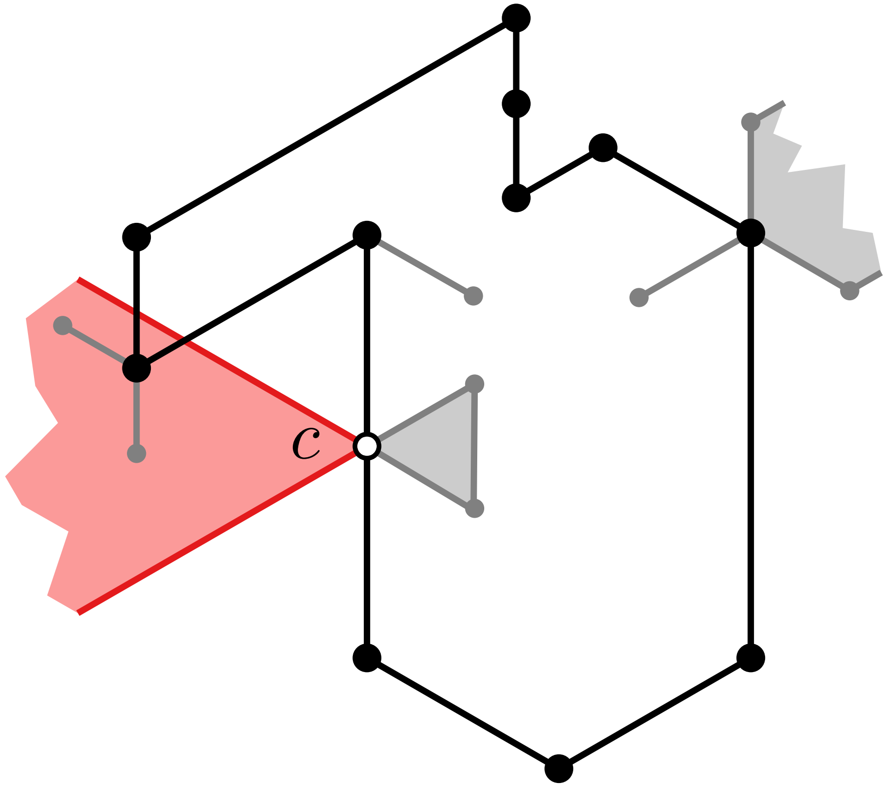

- Geometric realization



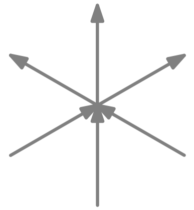
# Cactus digraphs – cycle block



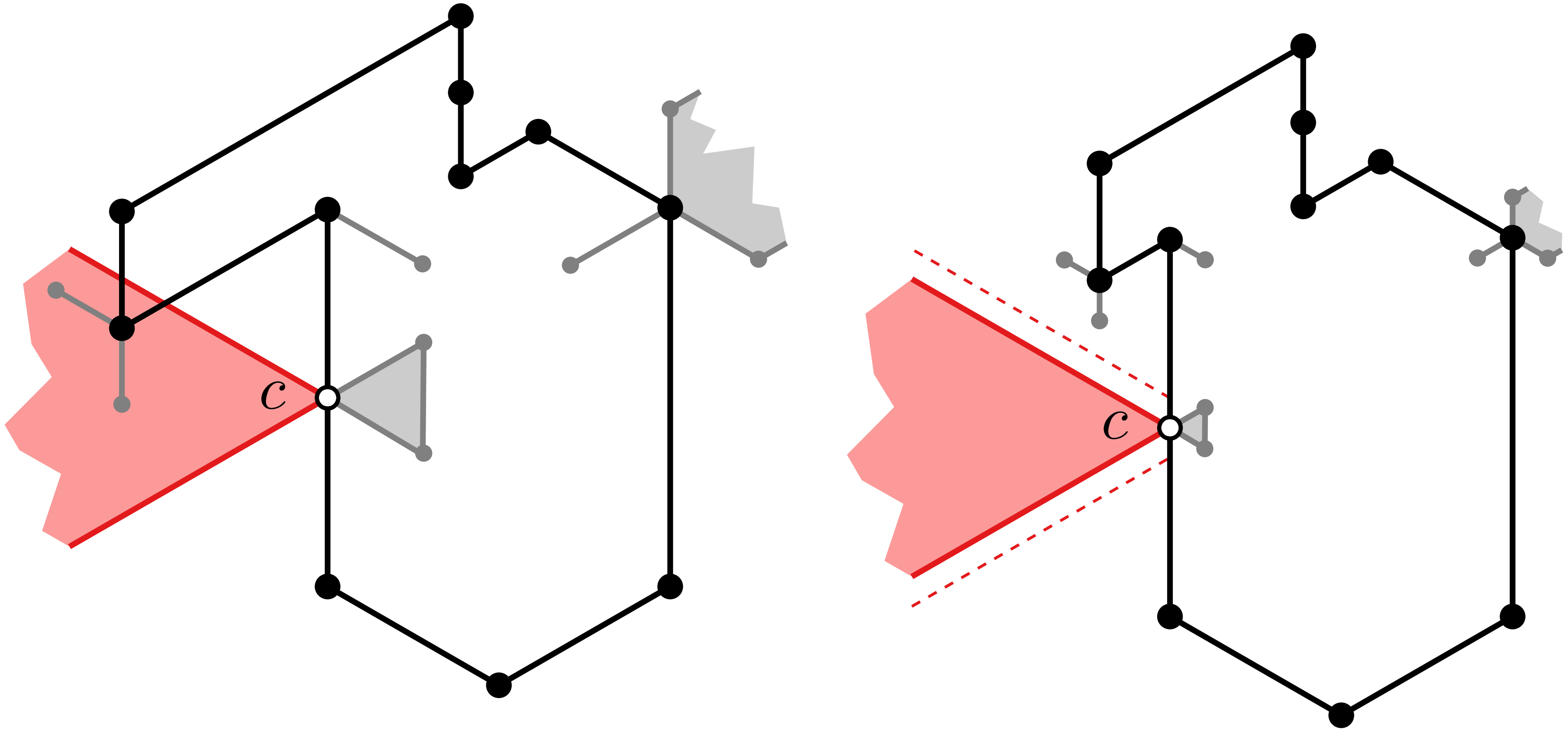

- Geometric realization



# Cactus digraphs – cycle block

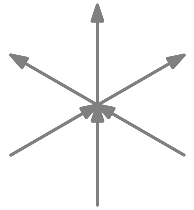



- Geometric realization

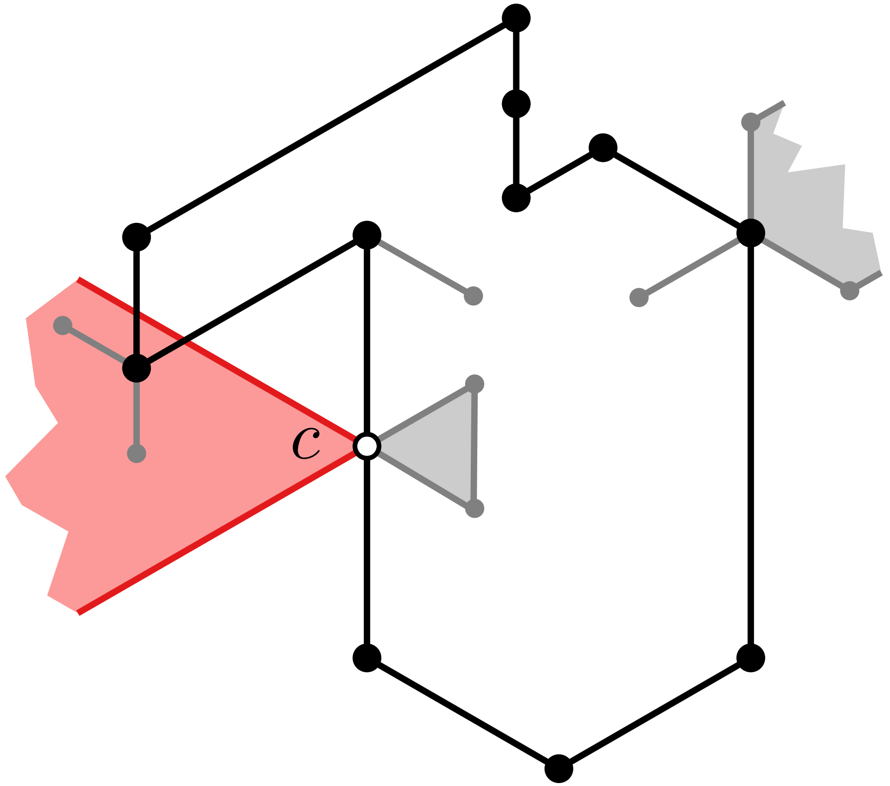


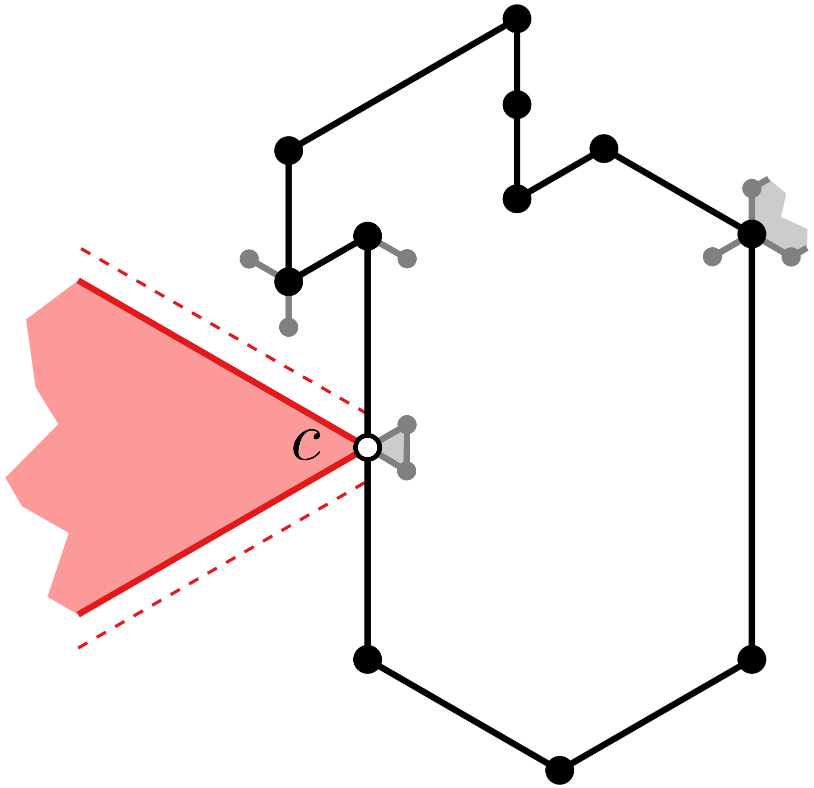

stay clear of parent
shrink exponentially

# Cactus digraphs – cycle block

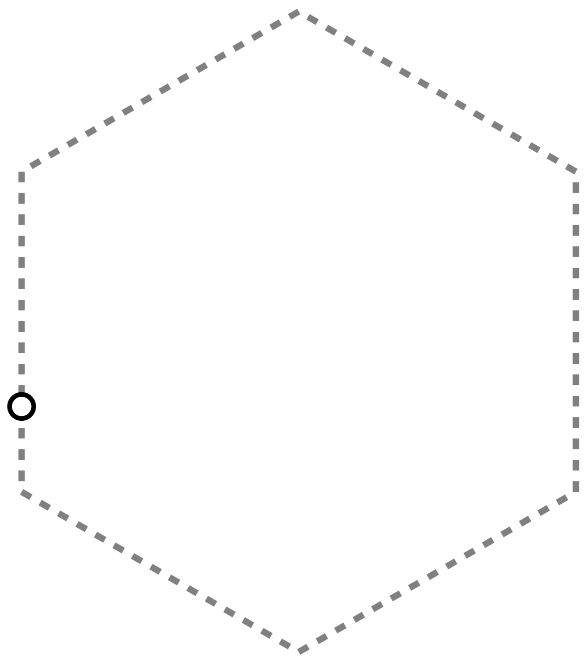



- Geometric realization



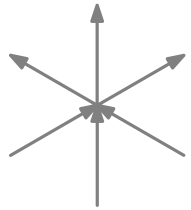




stay clear of parent
shrink exponentially

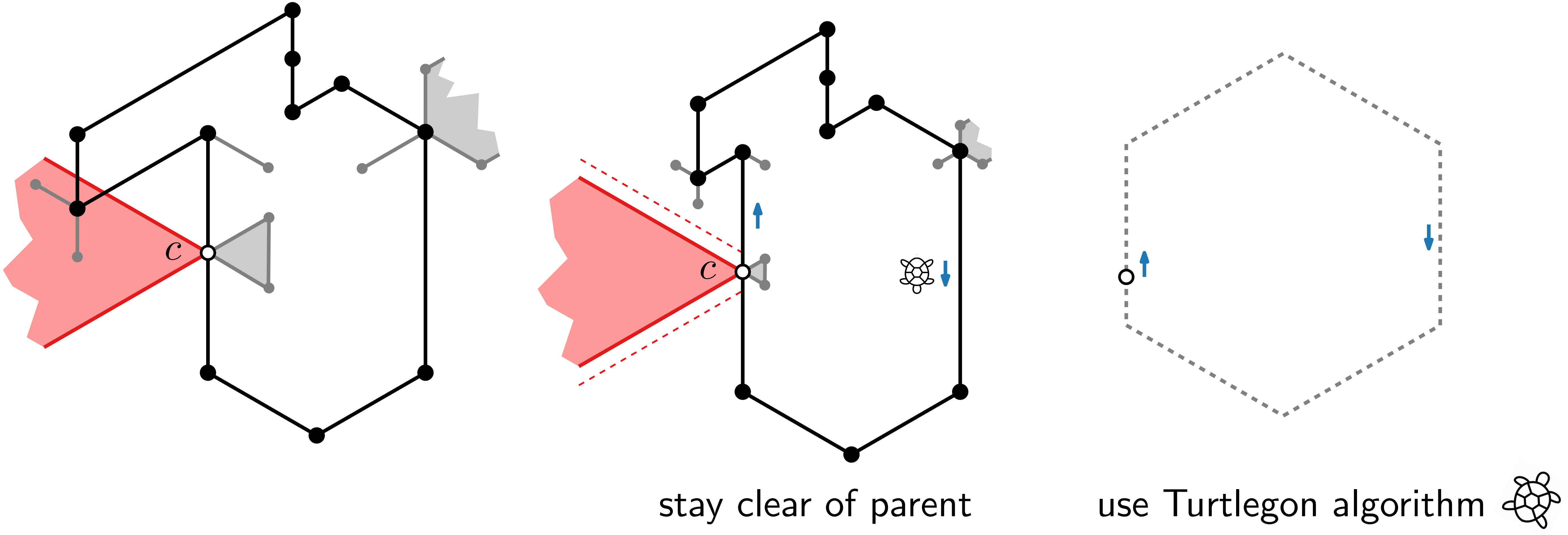


use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block



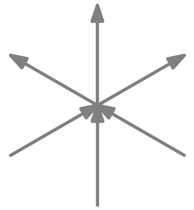

- Geometric realization



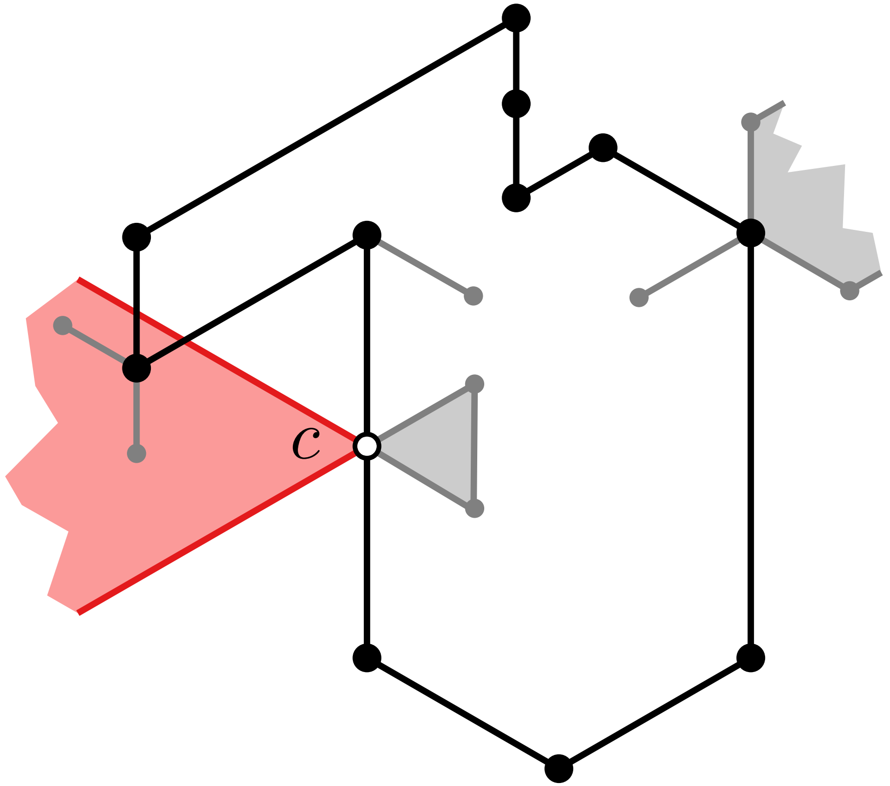

stay clear of parent
shrink exponentially

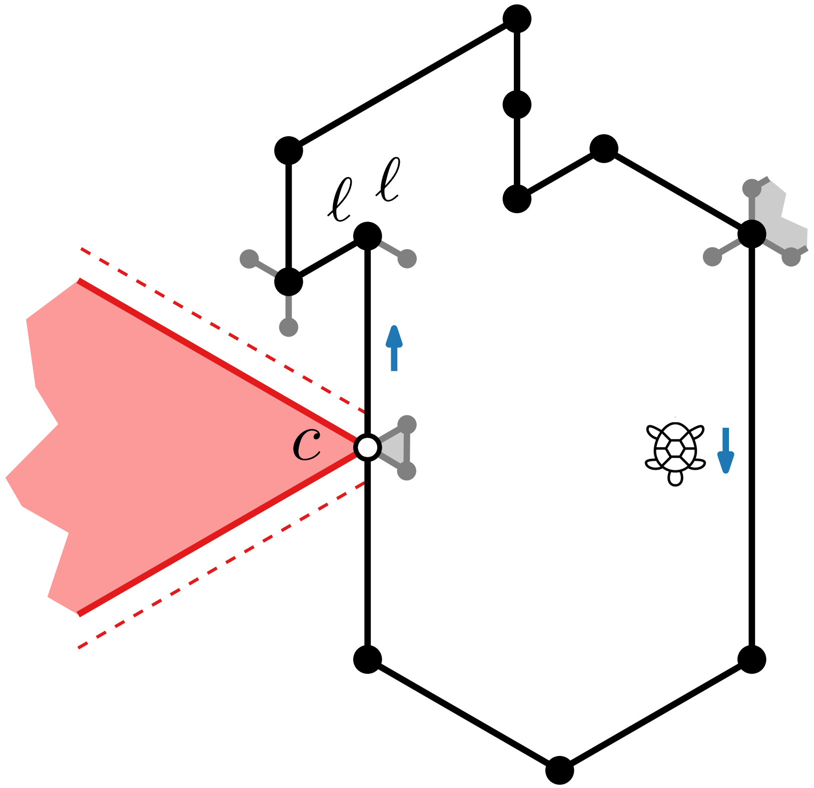
use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block

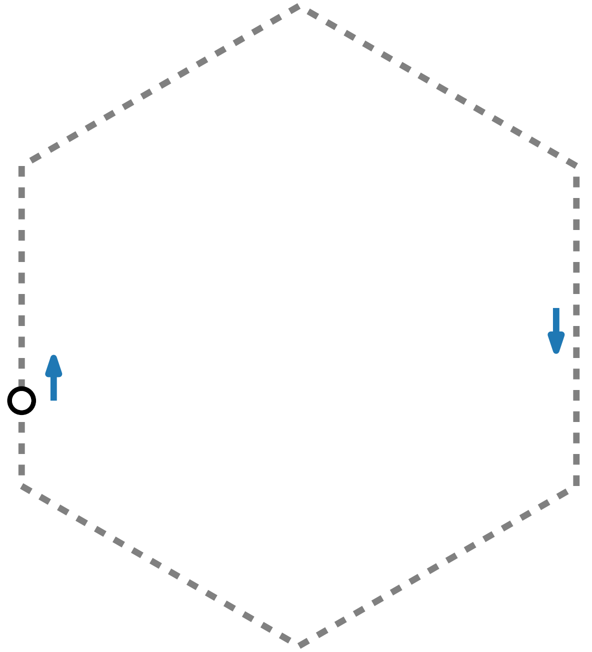



- Geometric realization



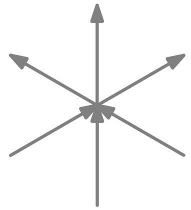




stay clear of parent
shrink exponentially

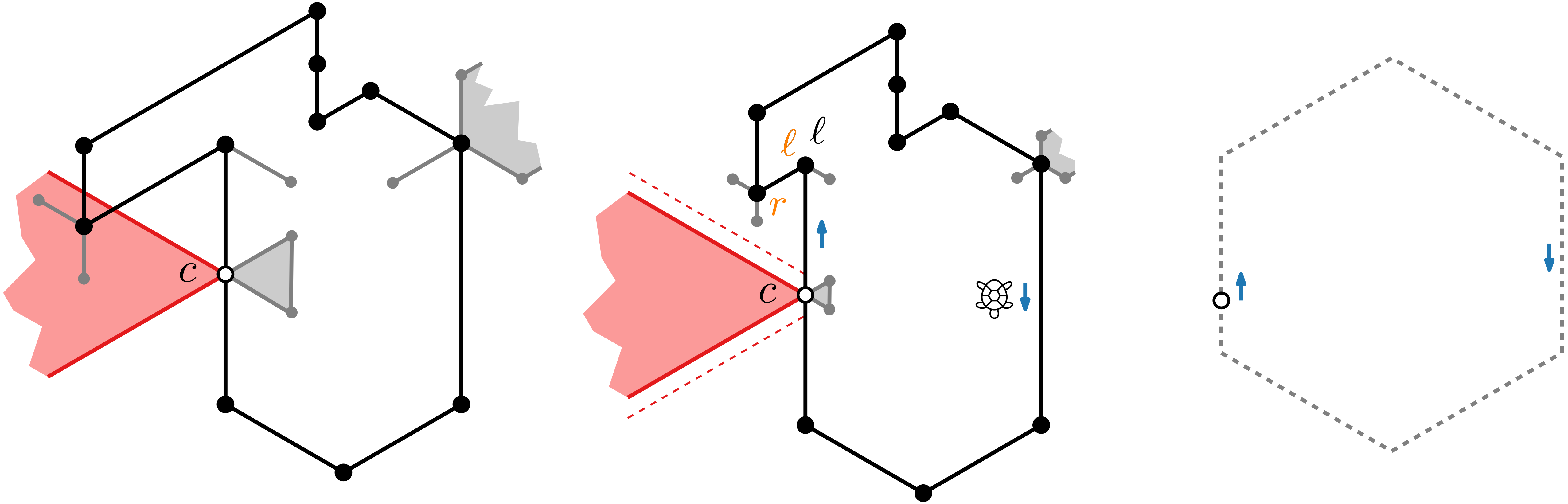


use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block



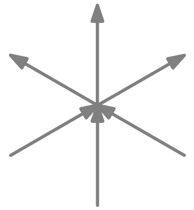

- Geometric realization




stay clear of parent
shrink exponentially

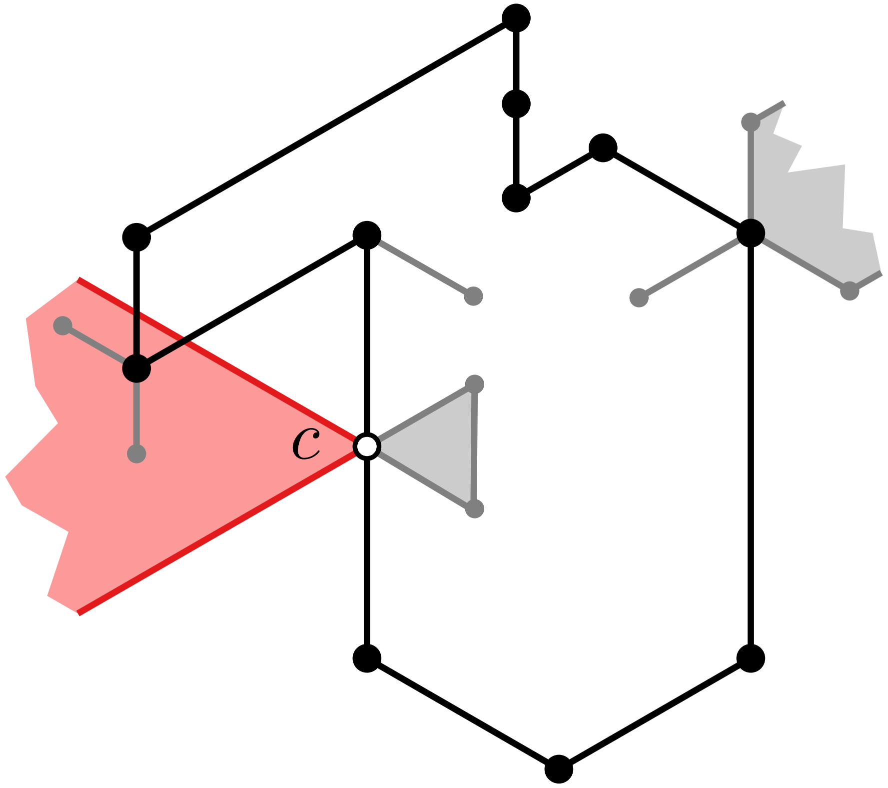
use Turtlegon algorithm
by Culberson and Rawlins

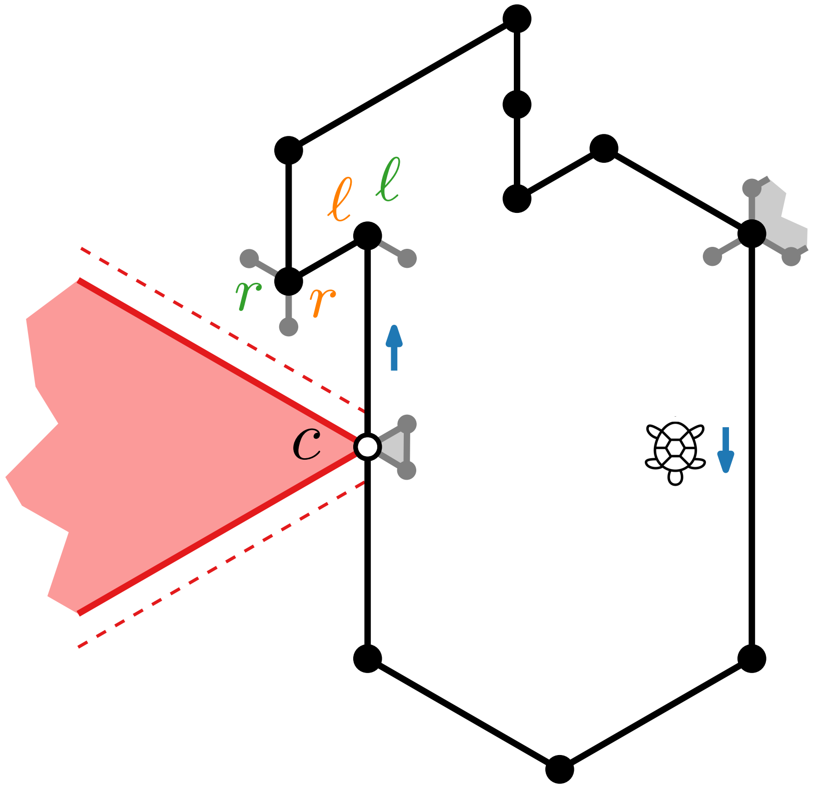
# Cactus digraphs – cycle block



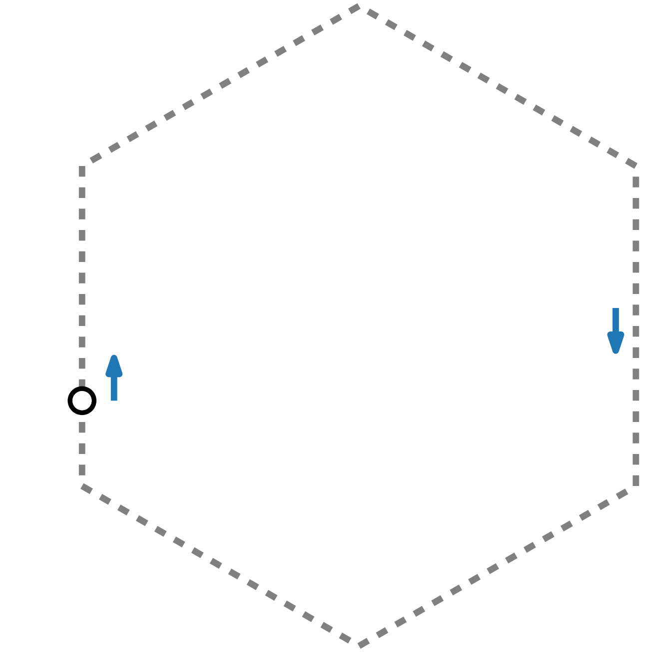
$k = 3$

- Geometric realization



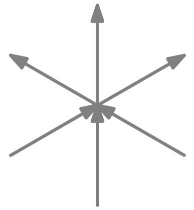




stay clear of parent
shrink exponentially

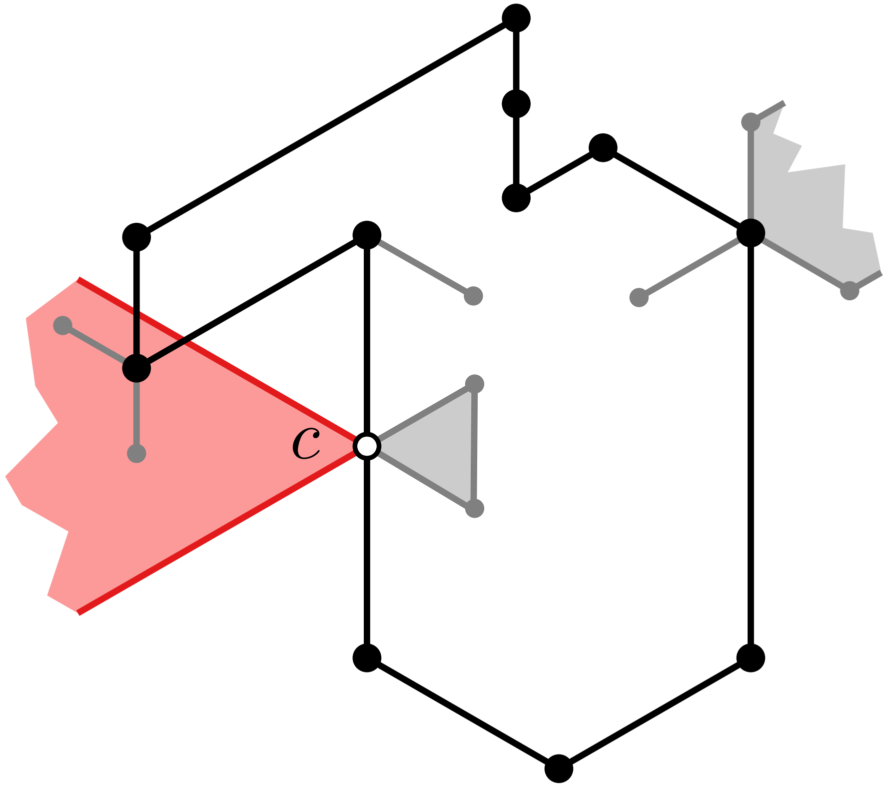


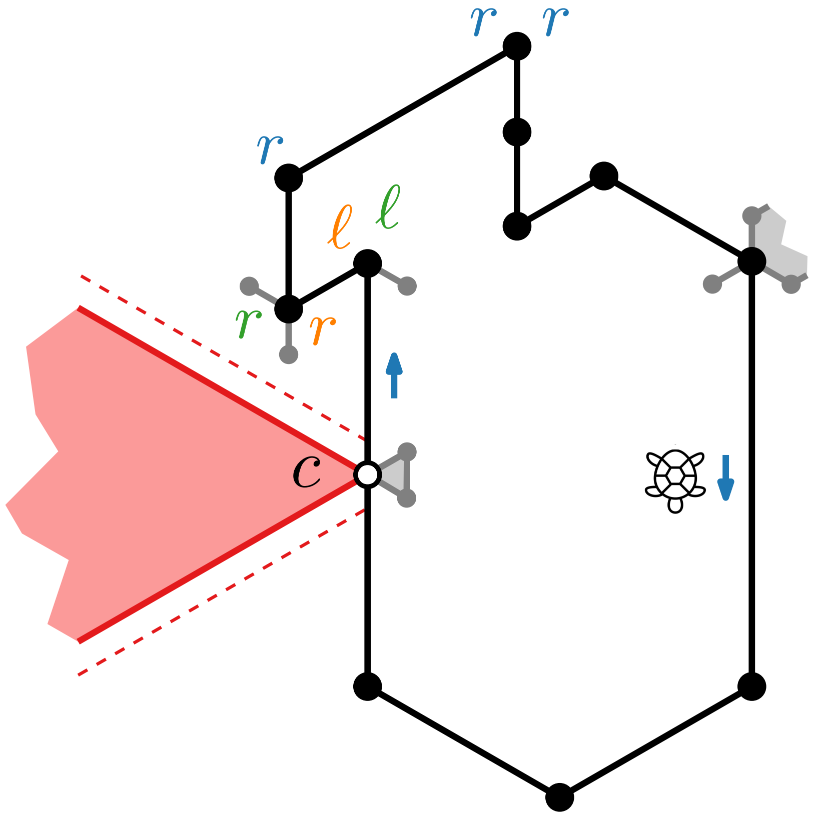
use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block

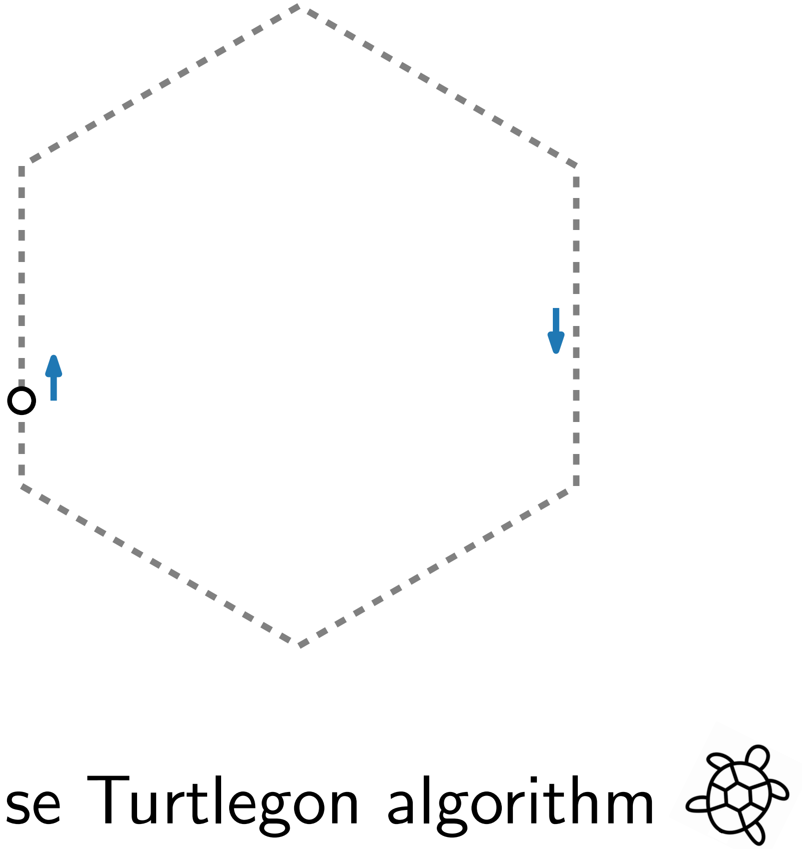



- Geometric realization



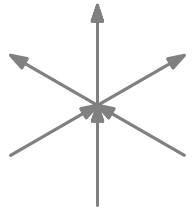




stay clear of parent
shrink exponentially



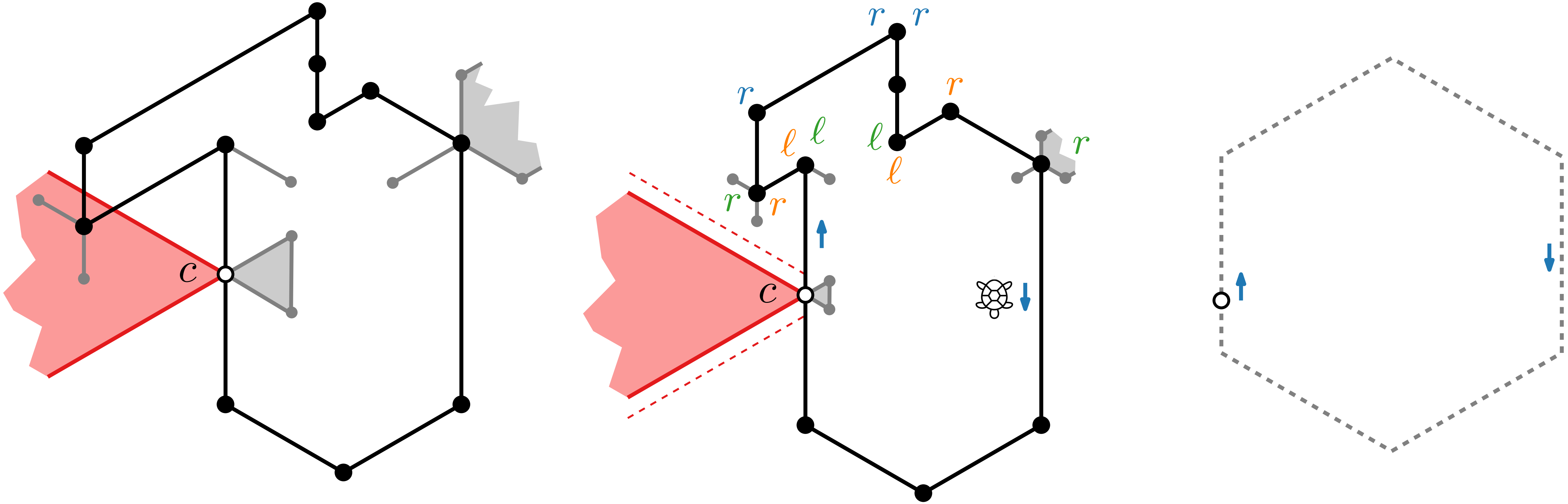
use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block



$k = 3$

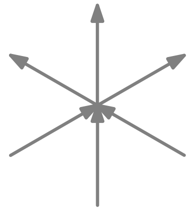
- Geometric realization




stay clear of parent
shrink exponentially

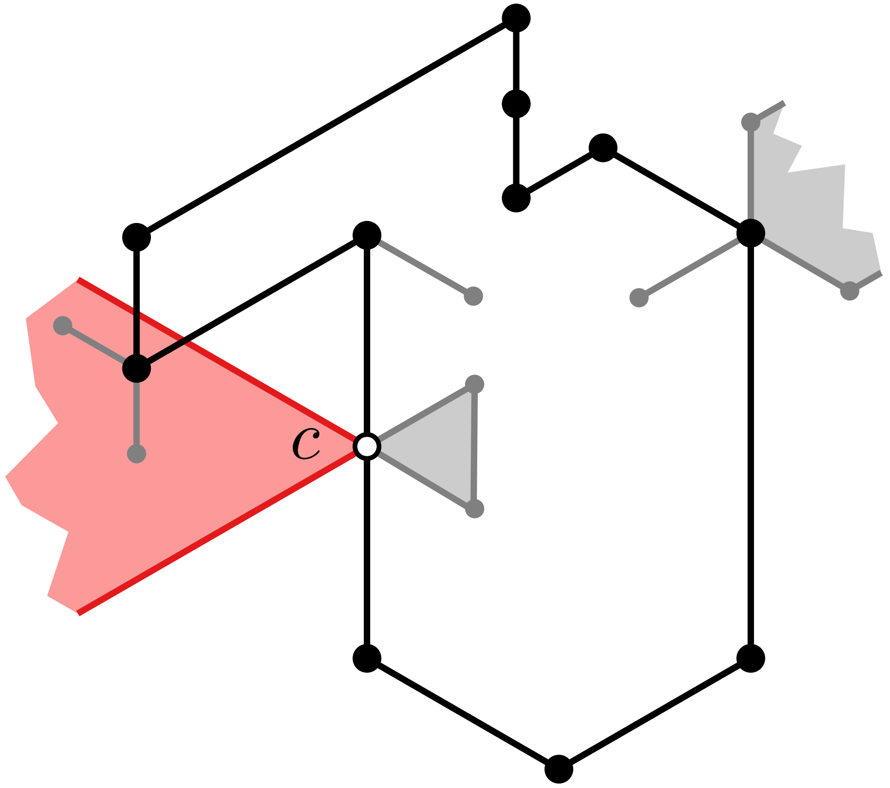
use Turtlegon algorithm
by Culberson and Rawlins

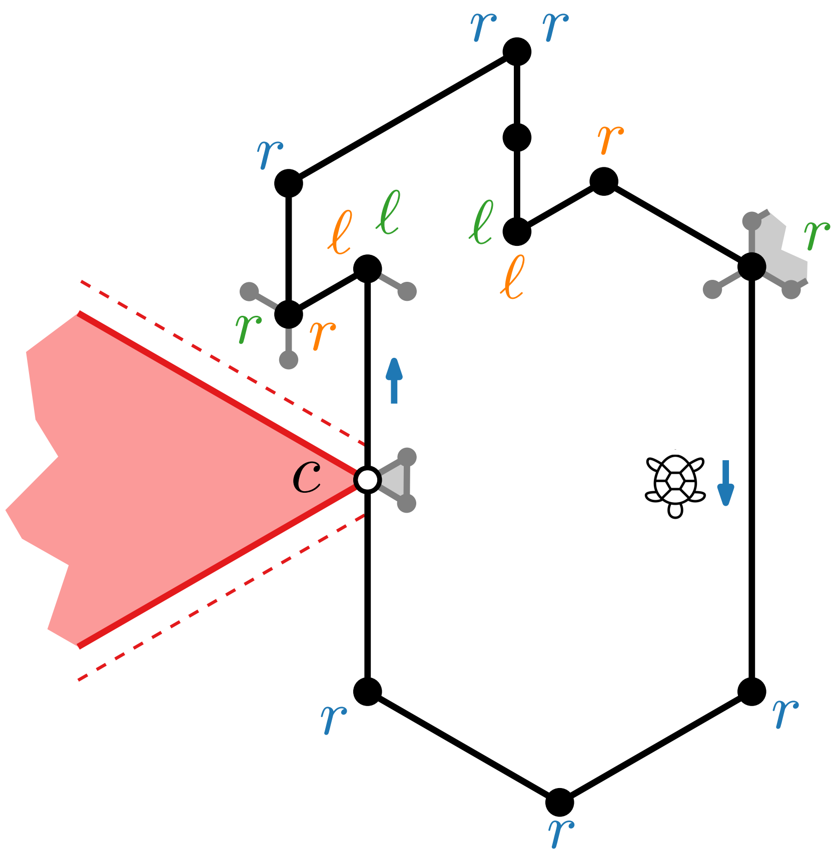
# Cactus digraphs – cycle block



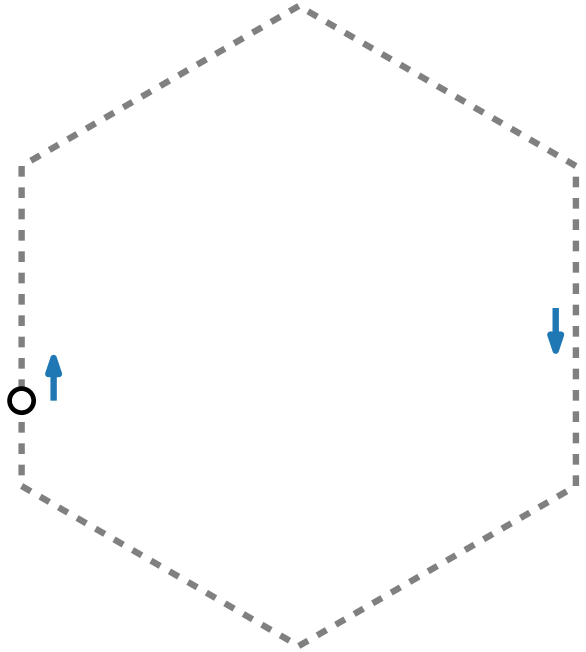
$k = 3$

- Geometric realization



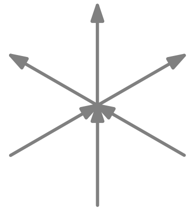




stay clear of parent
shrink exponentially



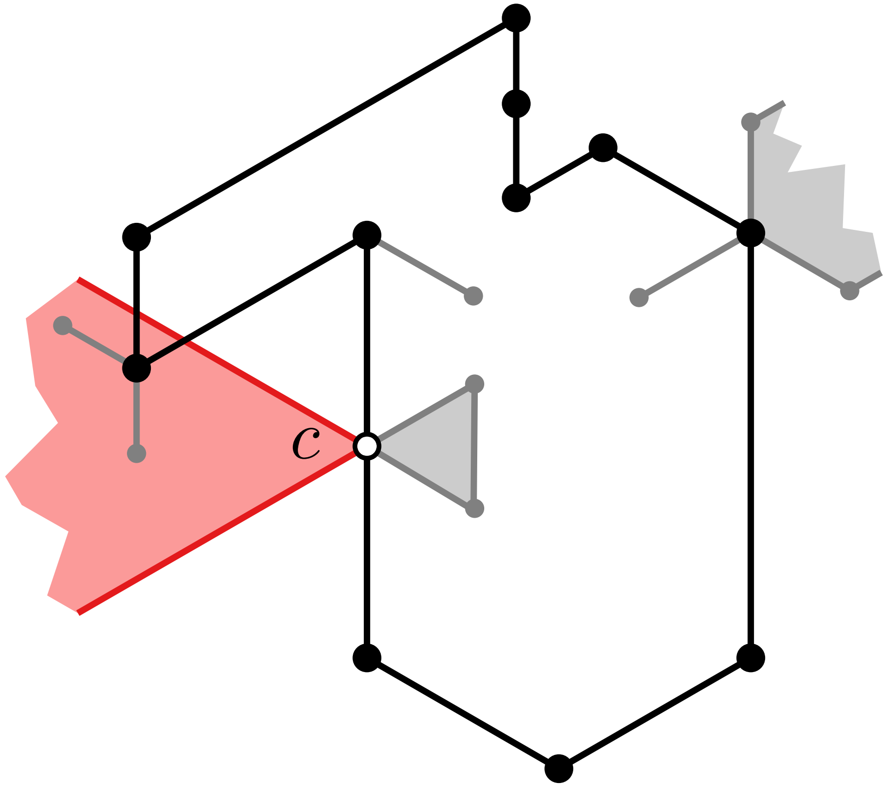
use Turtlegon algorithm
by Culberson and Rawlins

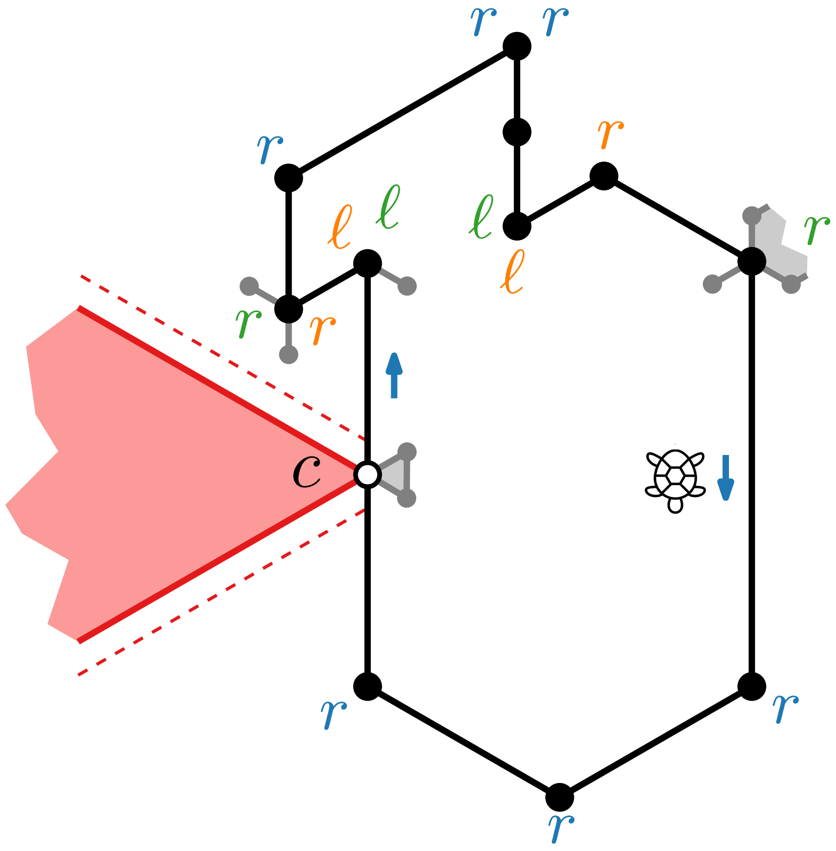
# Cactus digraphs – cycle block



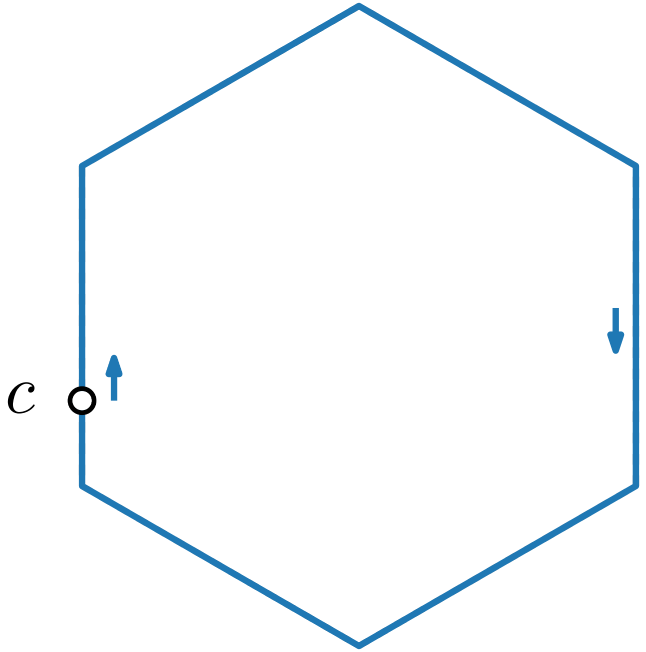
$k = 3$

- Geometric realization



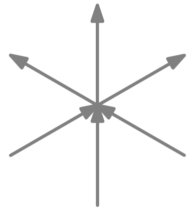




stay clear of parent
shrink exponentially



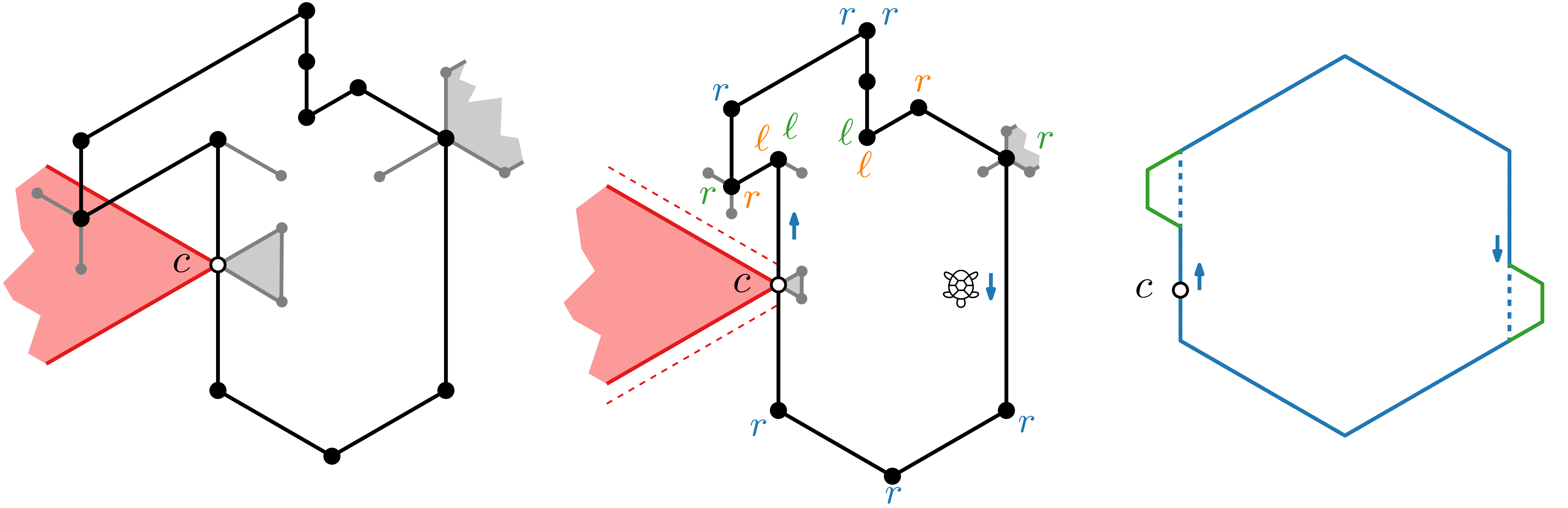

use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block



$k = 3$

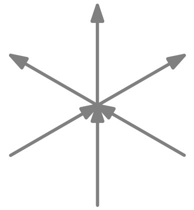
- Geometric realization



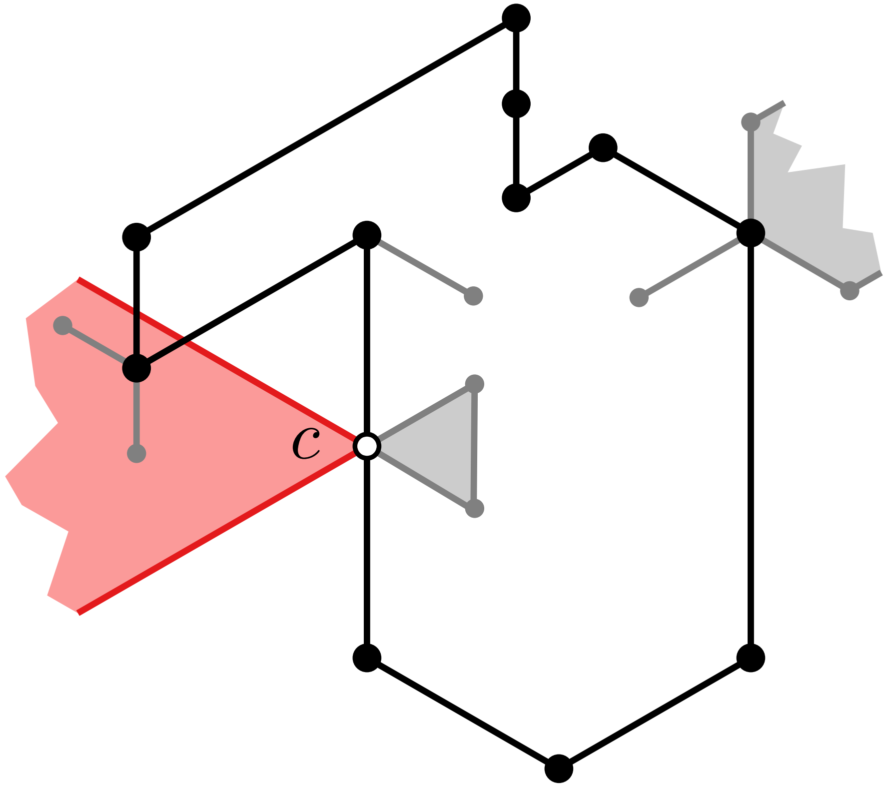

stay clear of parent
shrink exponentially

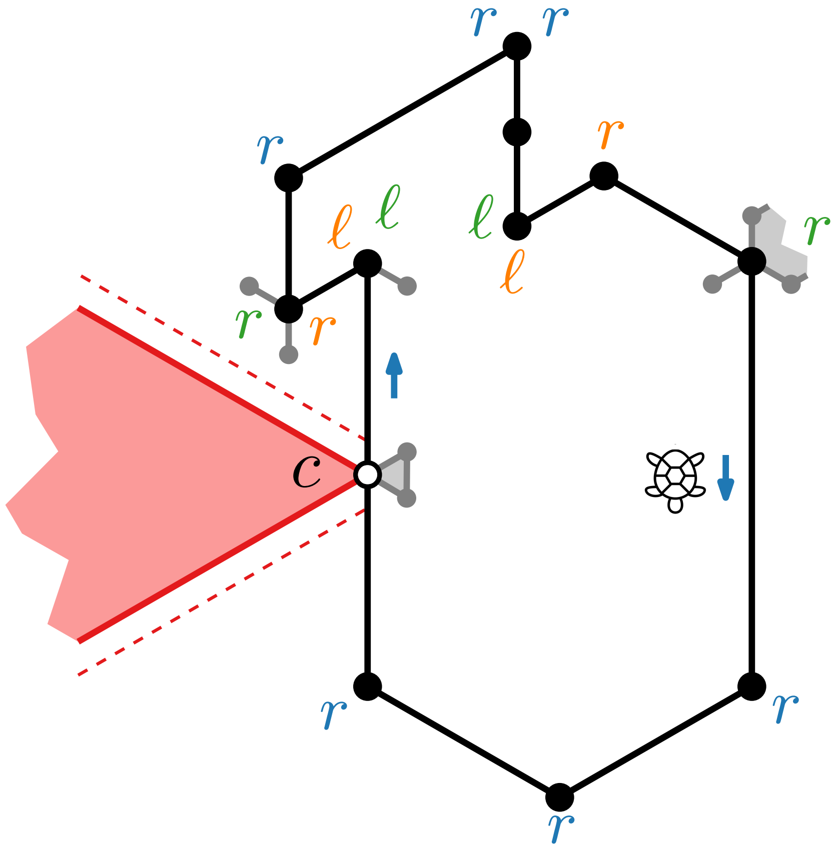
use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block

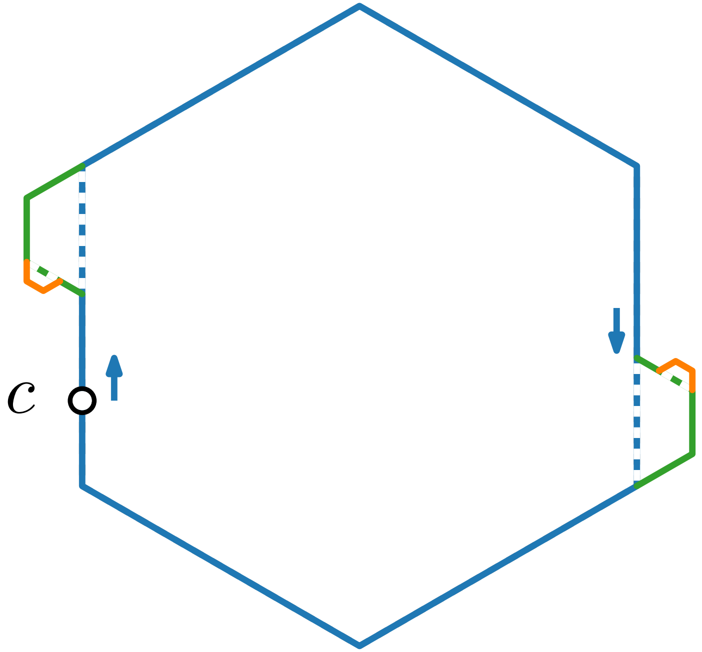



- Geometric realization



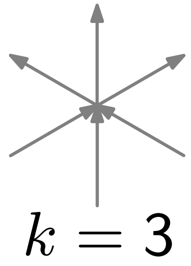




stay clear of parent
shrink exponentially

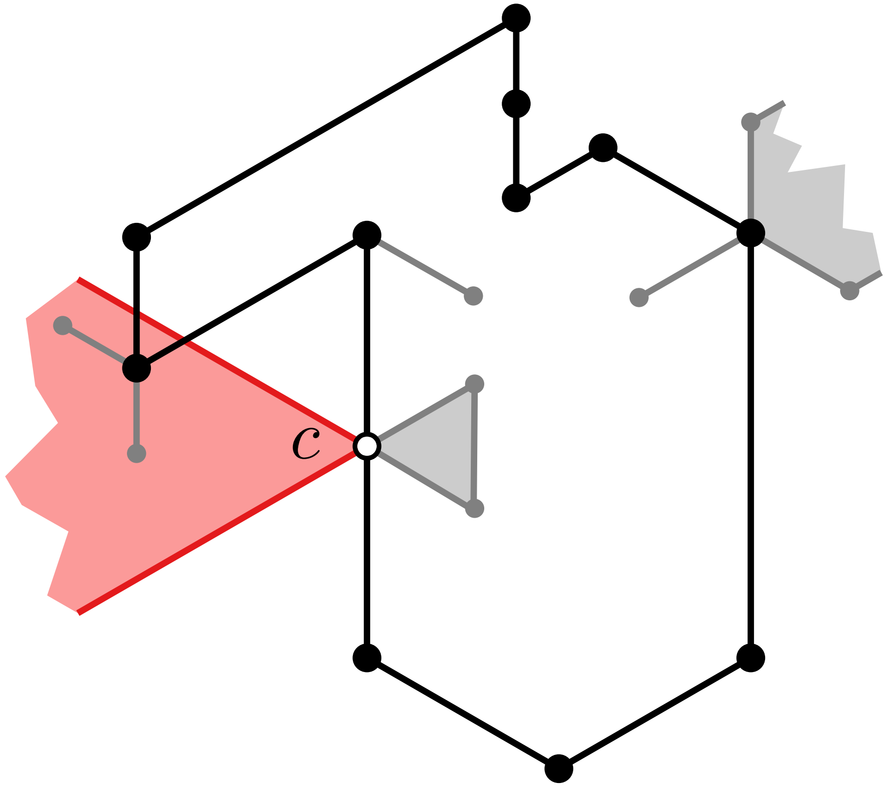


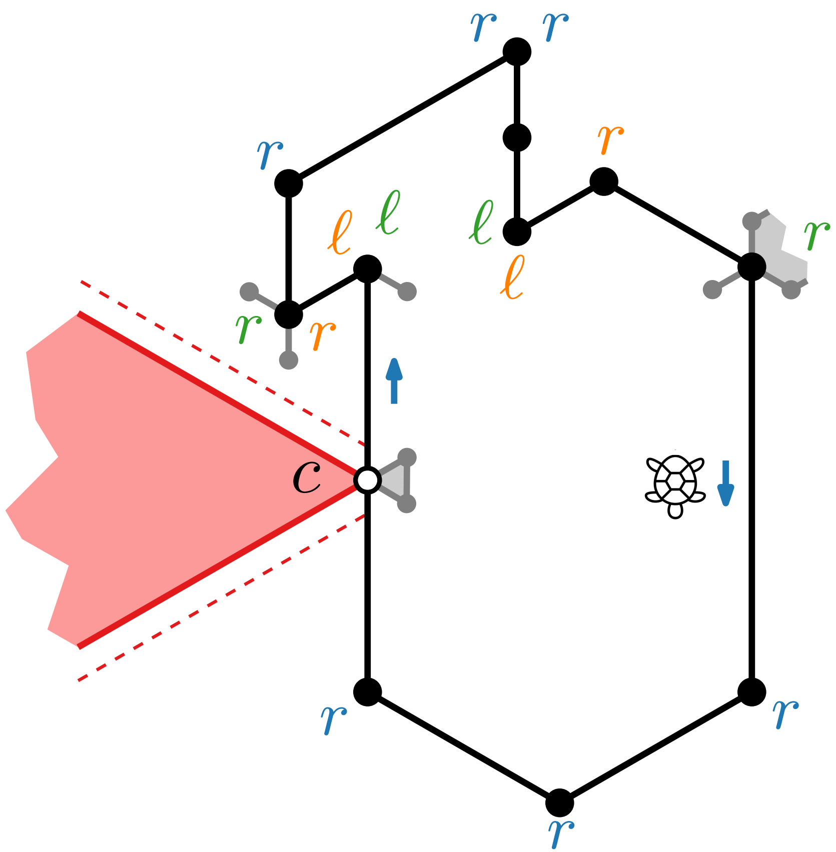

use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block

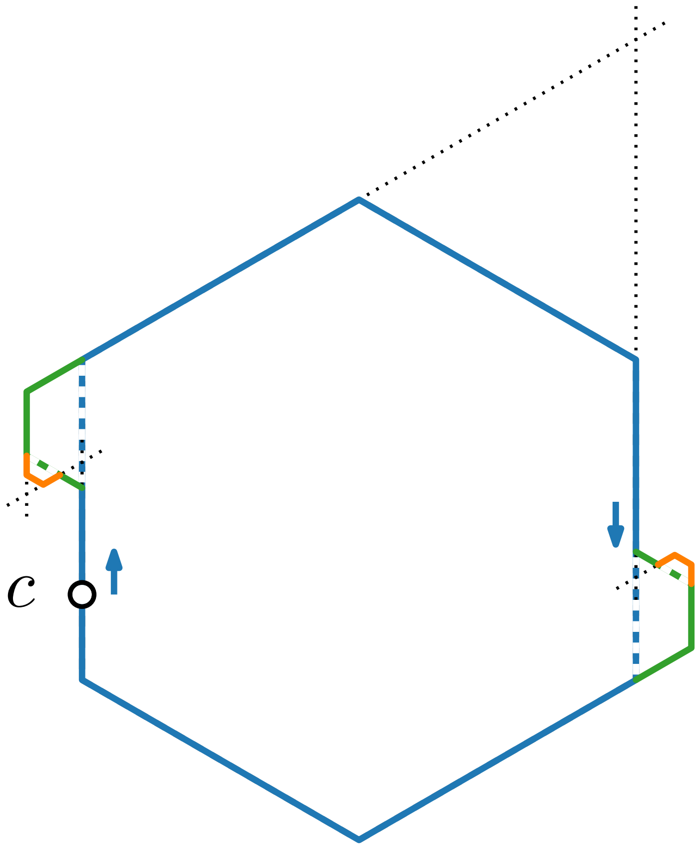



- Geometric realization



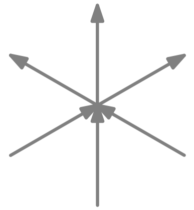




stay clear of parent
shrink exponentially

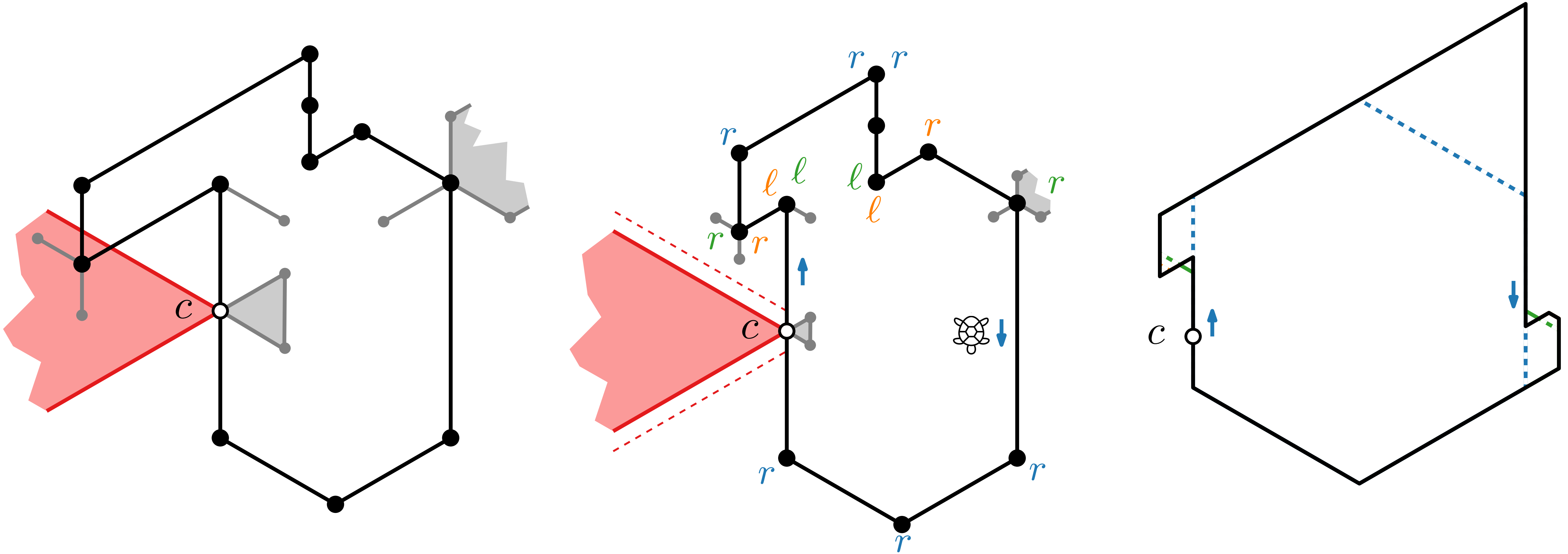



use Turtlegon algorithm
by Culberson and Rawlins

# Cactus digraphs – cycle block




- Geometric realization




stay clear of parent
shrink exponentially

use Turtlegon algorithm
by Culberson and Rawlins

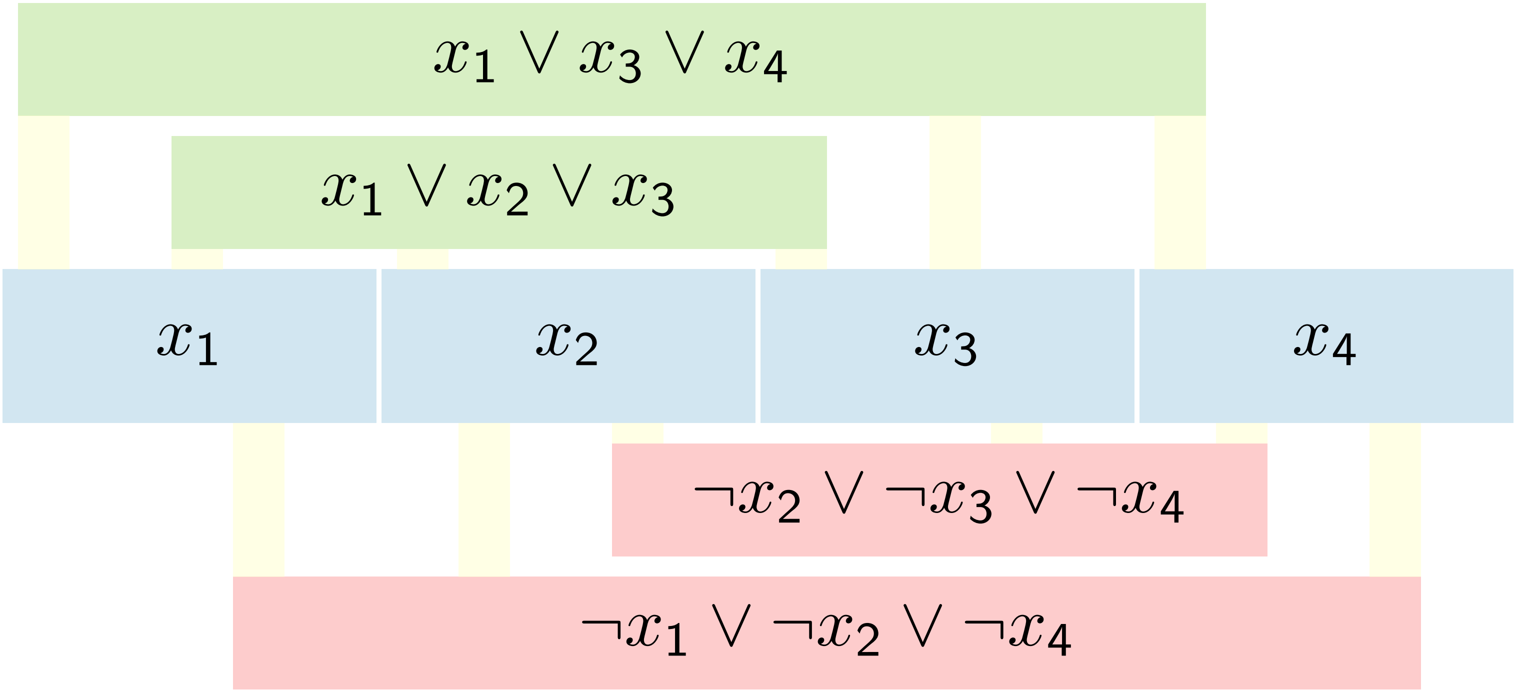
# Outerplanar digraphs

**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

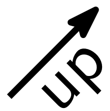
# Outerplanar digraphs

**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

Reduction from PLANAR MONTONE 3-SAT:

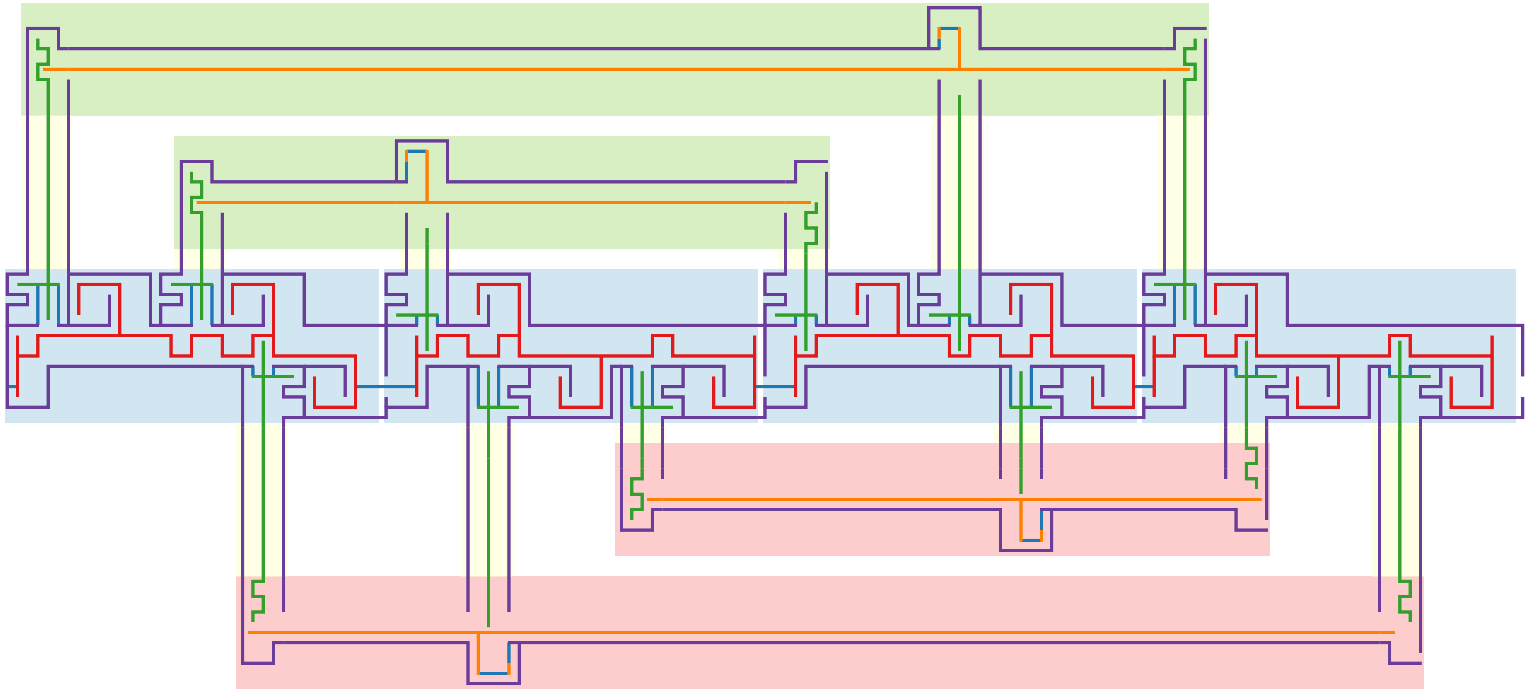


# Outerplanar digraphs

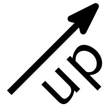



**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

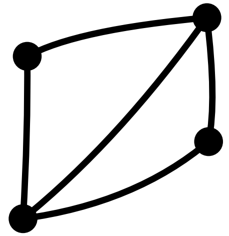
Reduction from PLANAR MONTONE 3-SAT:

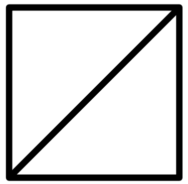


# Outerplanar digraphs

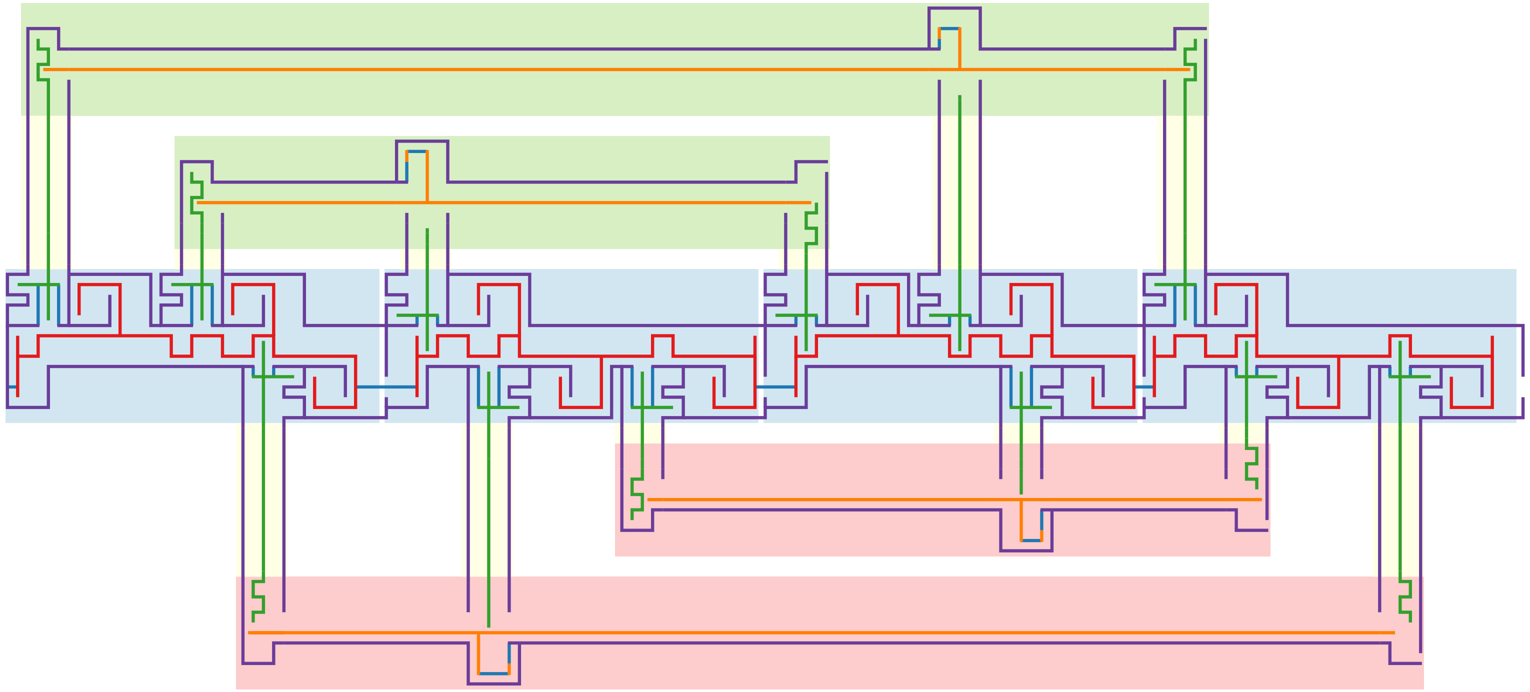



**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

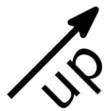




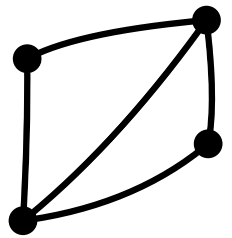
Reduction from PLANAR MONTONE 3-SAT:

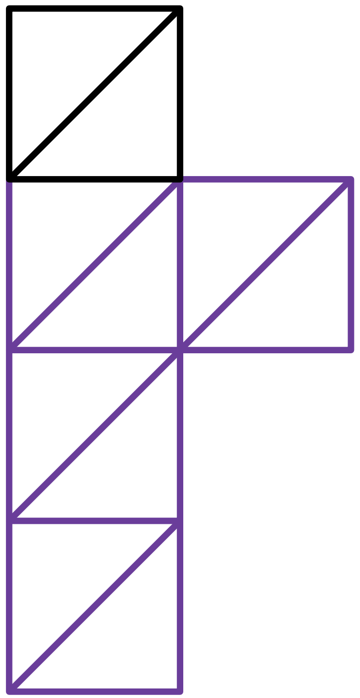


# Outerplanar digraphs

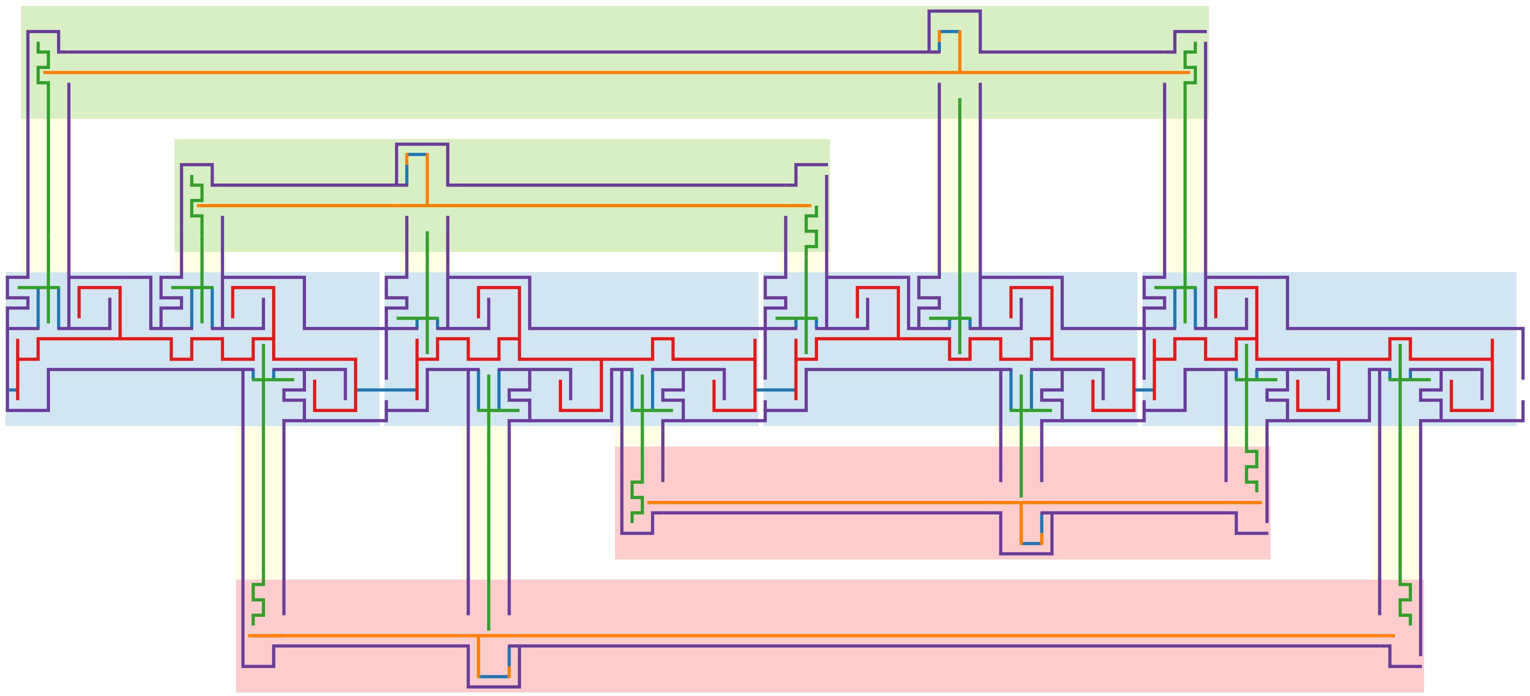



> **Theorem.**
> Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

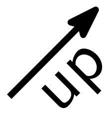




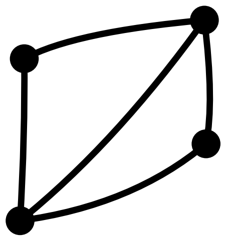
Reduction from PLANAR MONTONE 3-SAT:

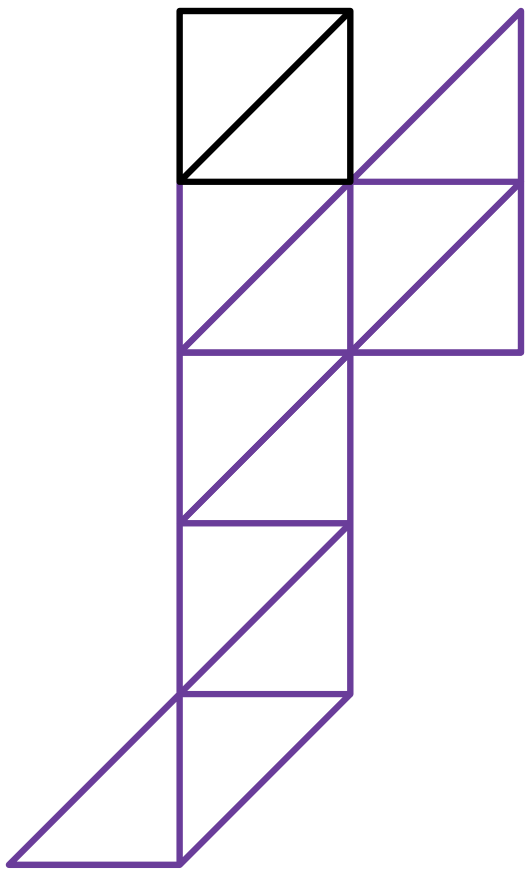


# Outerplanar digraphs

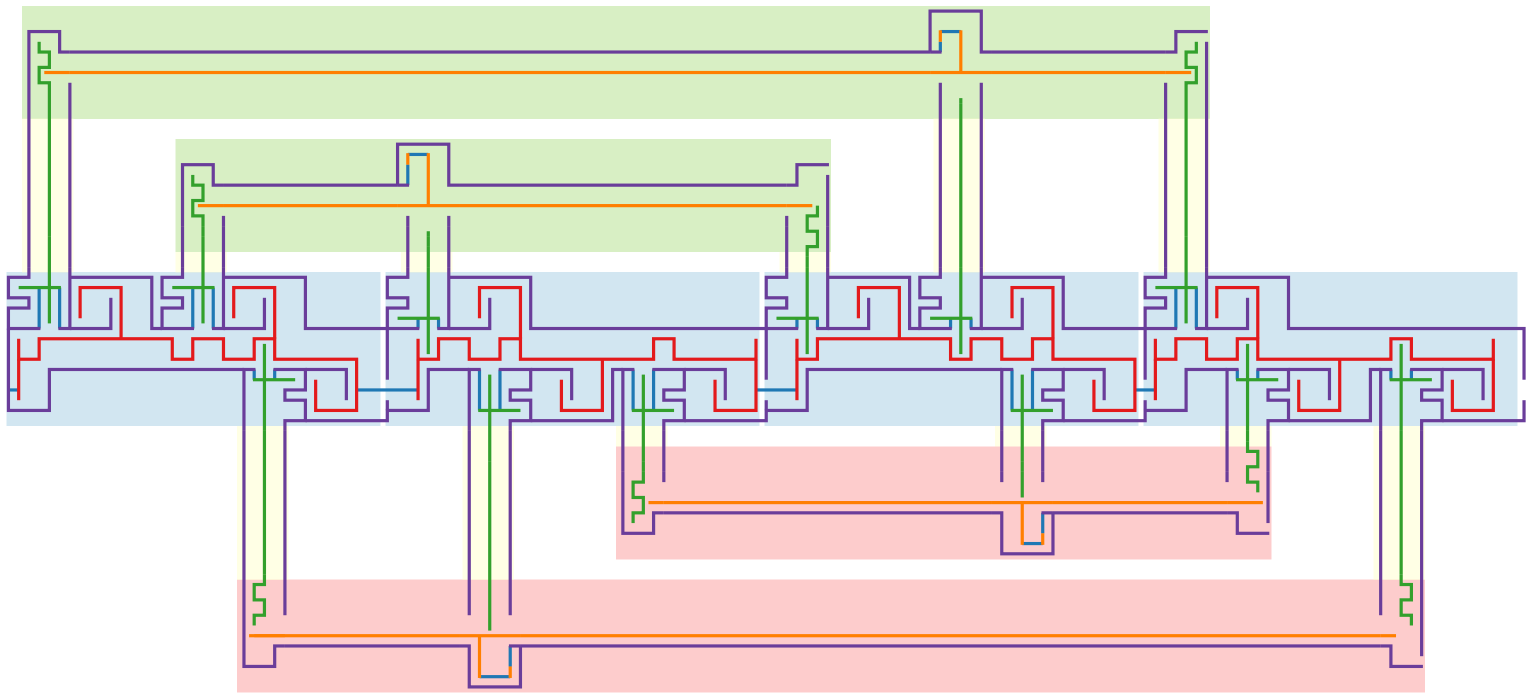



**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

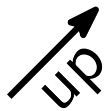




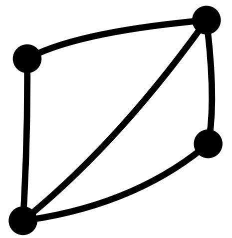
Reduction from PLANAR MONTONE 3-SAT:

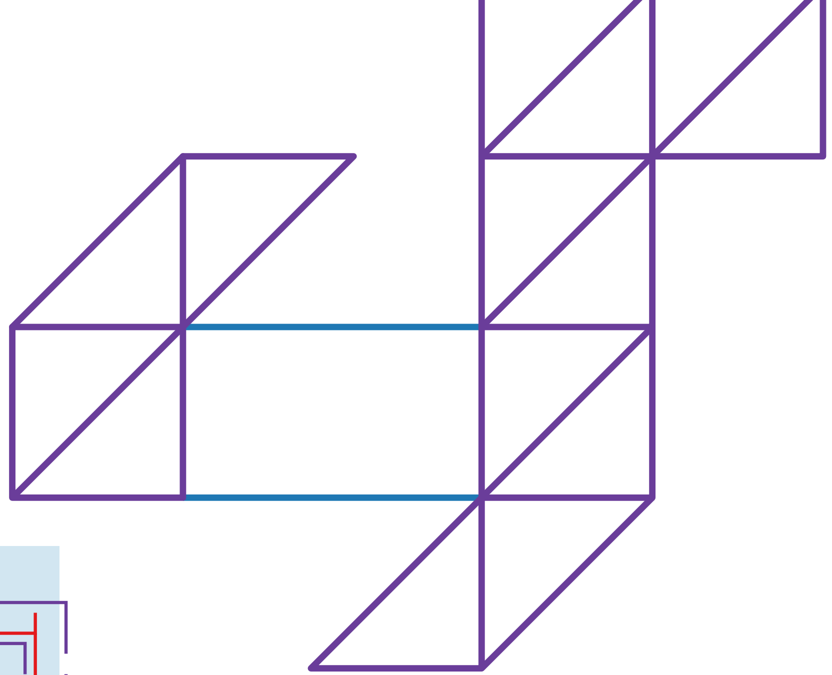


# Outerplanar digraphs

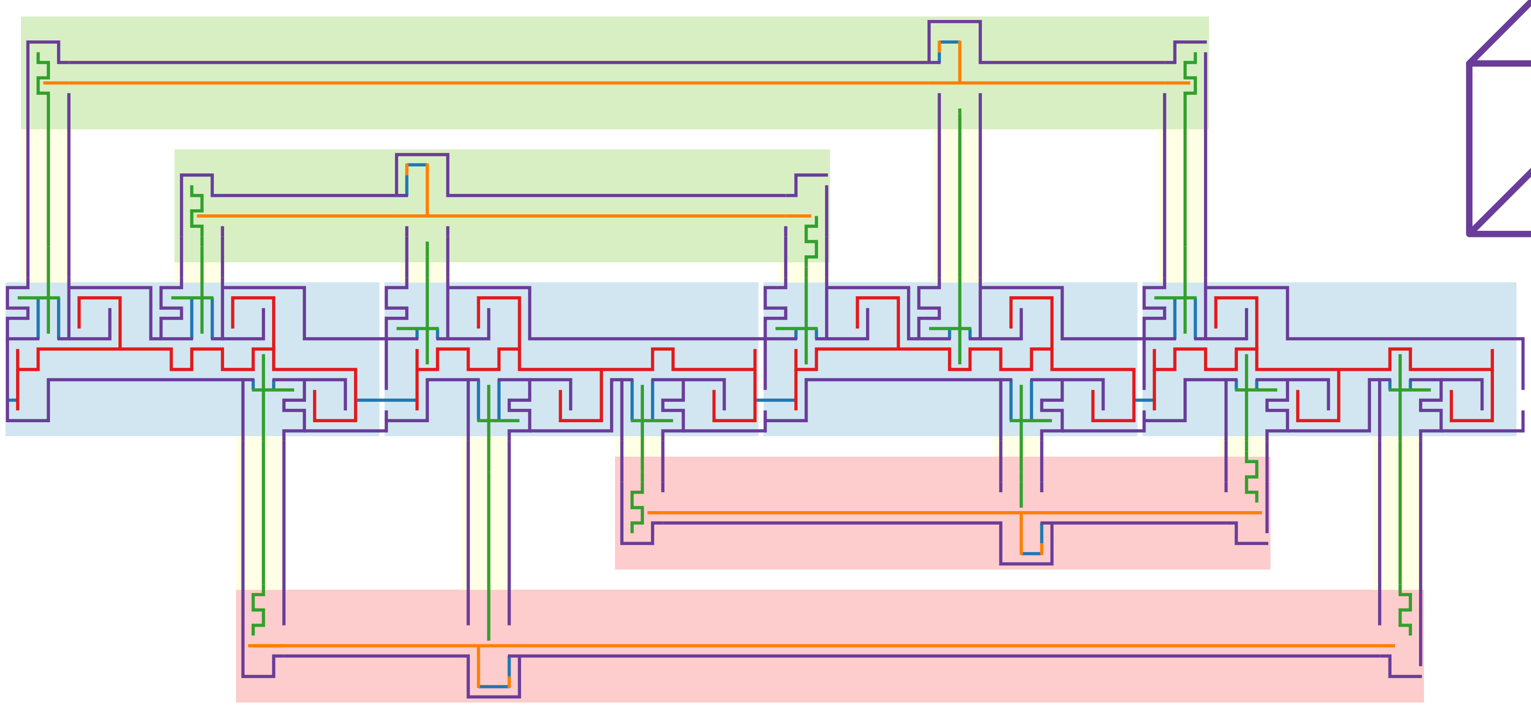



**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

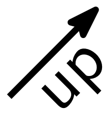




Reduction from PLANAR MONTONE 3-SAT:

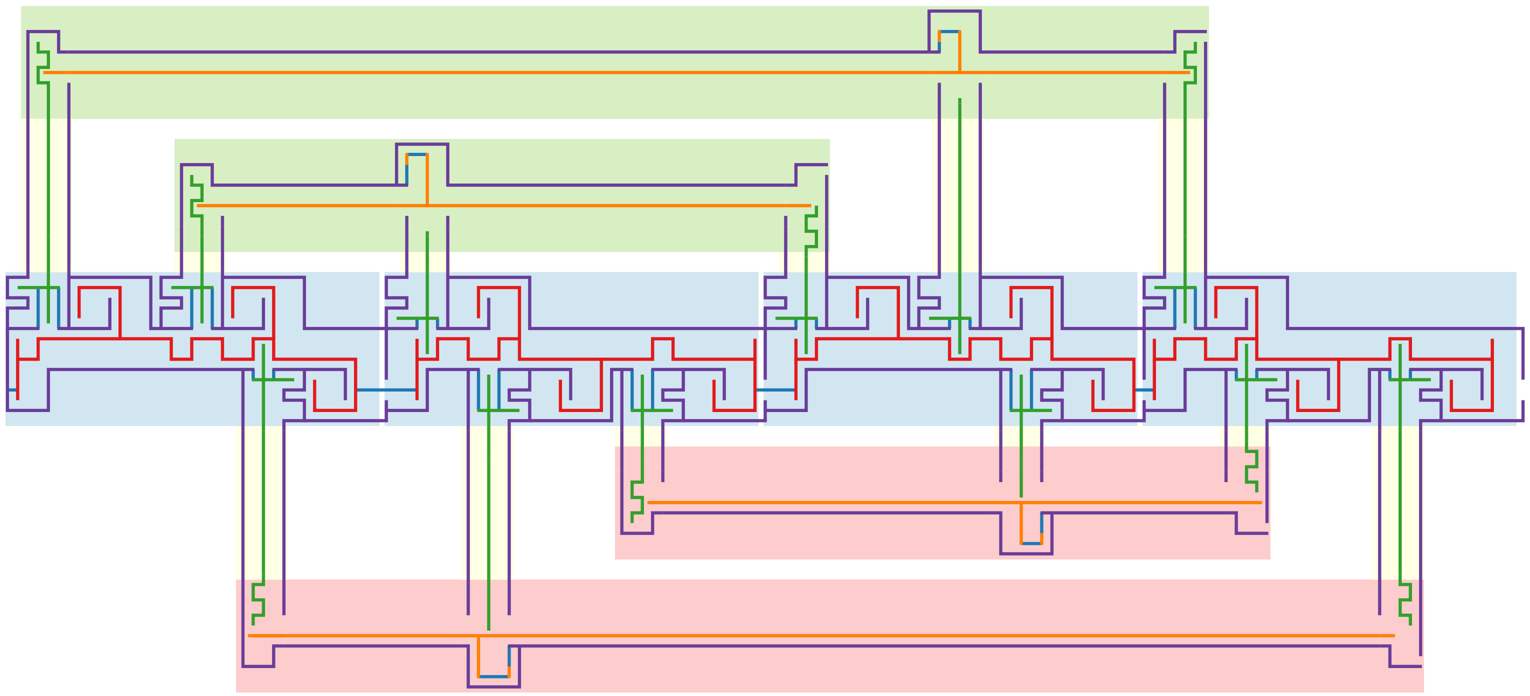


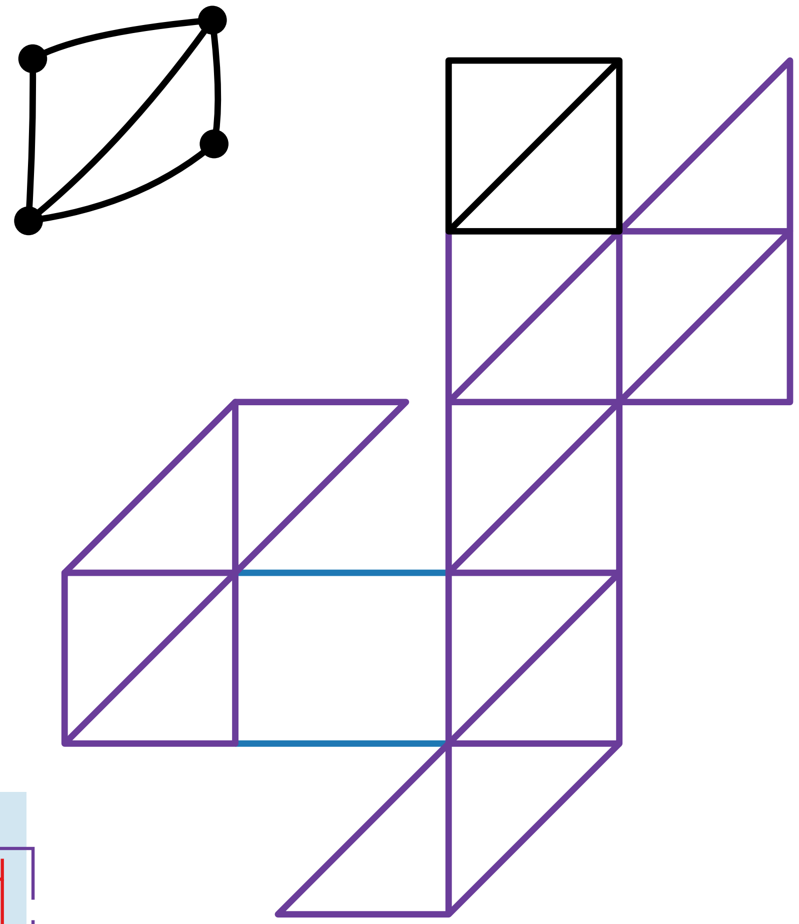
# Outerplanar digraphs




**Theorem.**
Given an outerplanar digraph $G$ with/without upward planar embedding, it is **NP-hard** to decide if $G$ admits an upward planar 3-slope drawing.

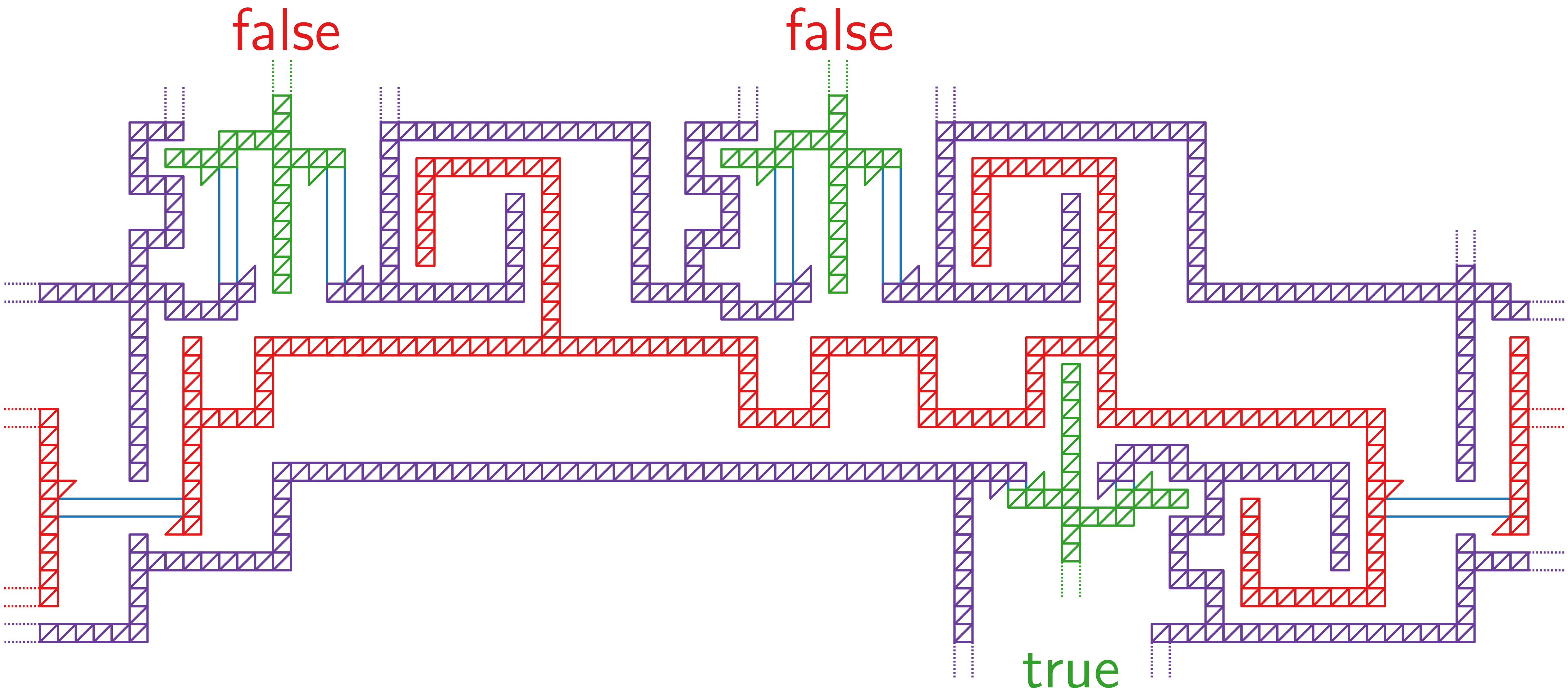
Reduction from Planar Montone 3-SAT:





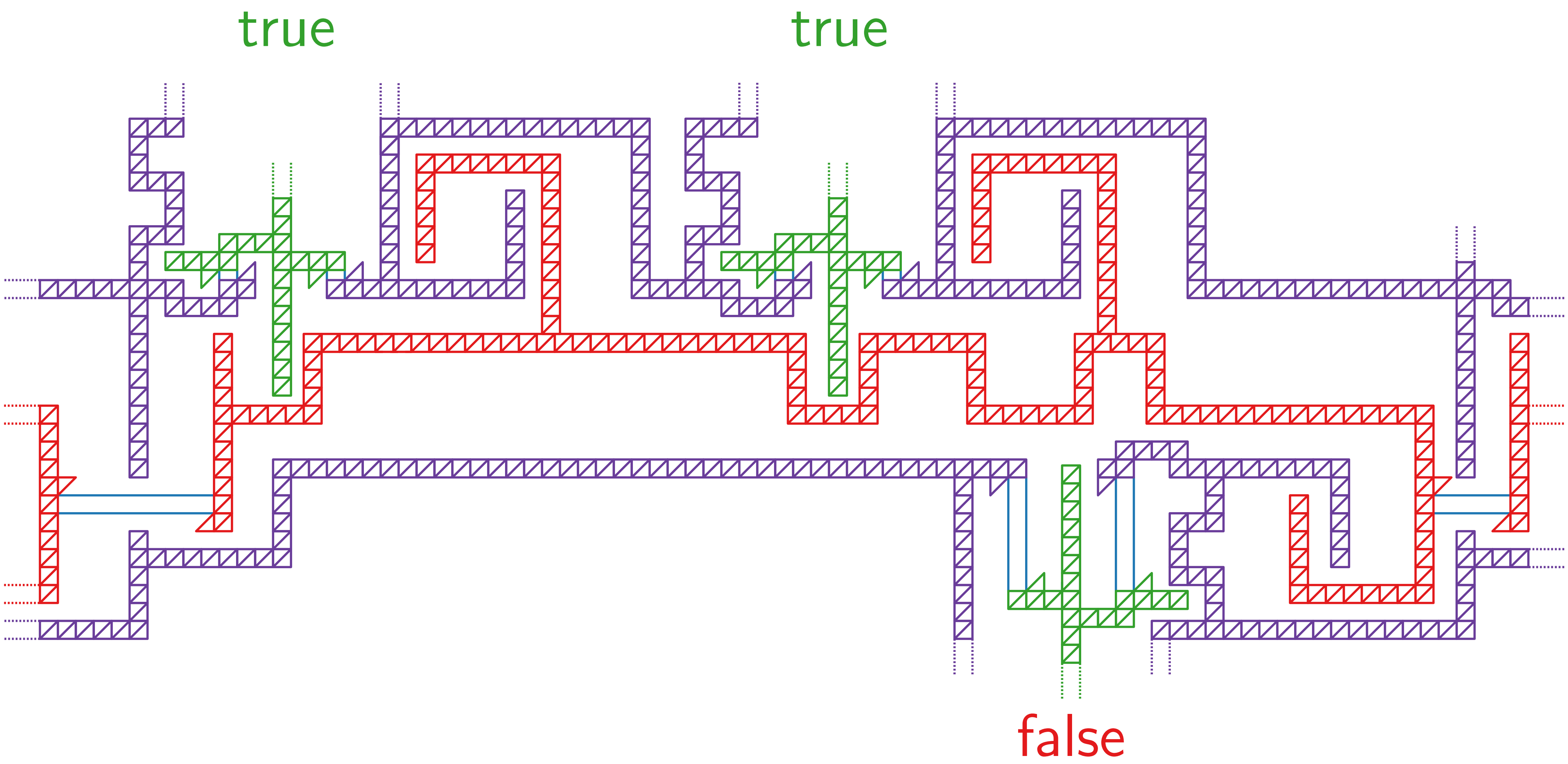
# Outerplanar digraphs

Variable gadget:



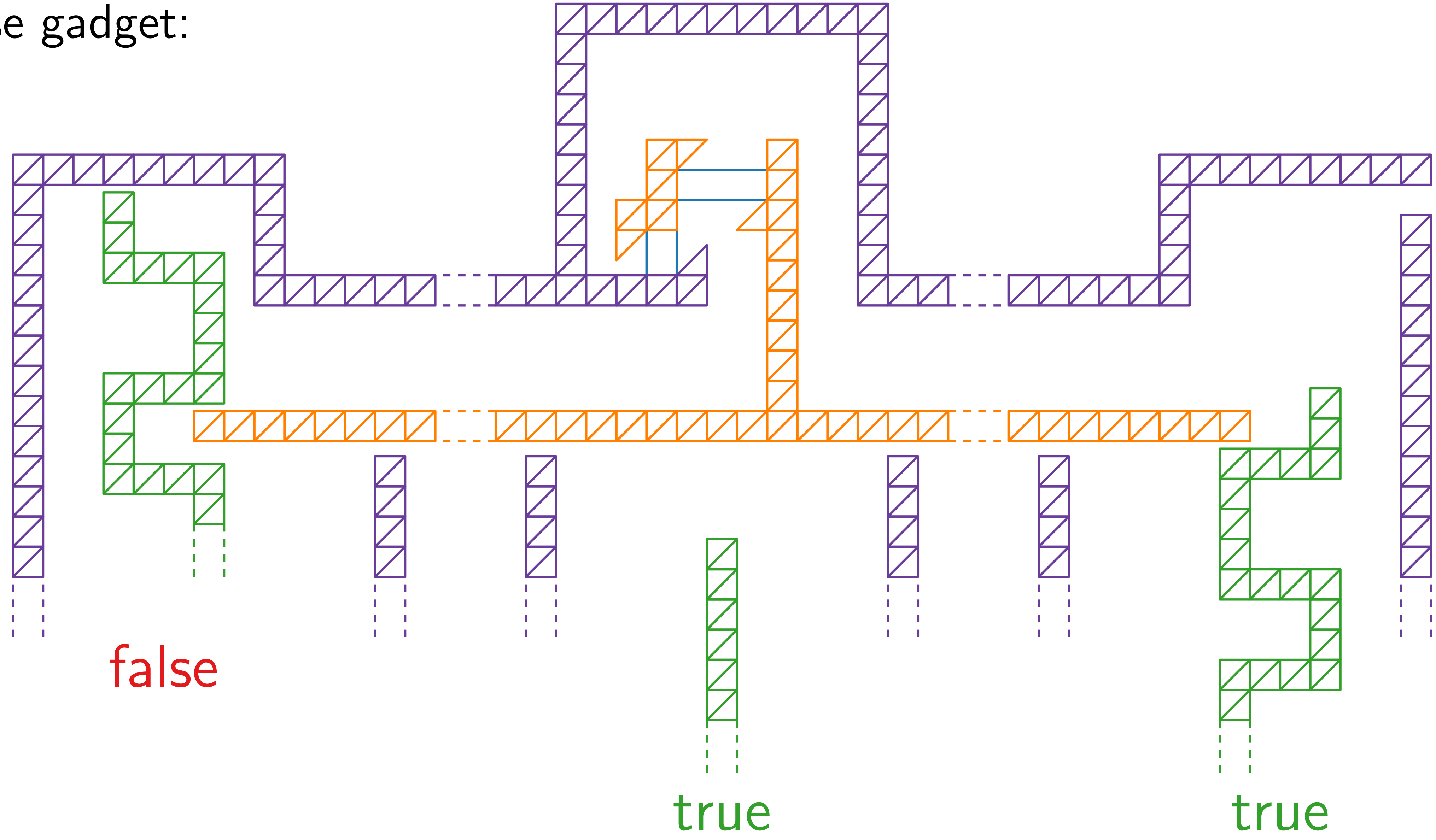
# Outerplanar digraphs

Variable gadget:



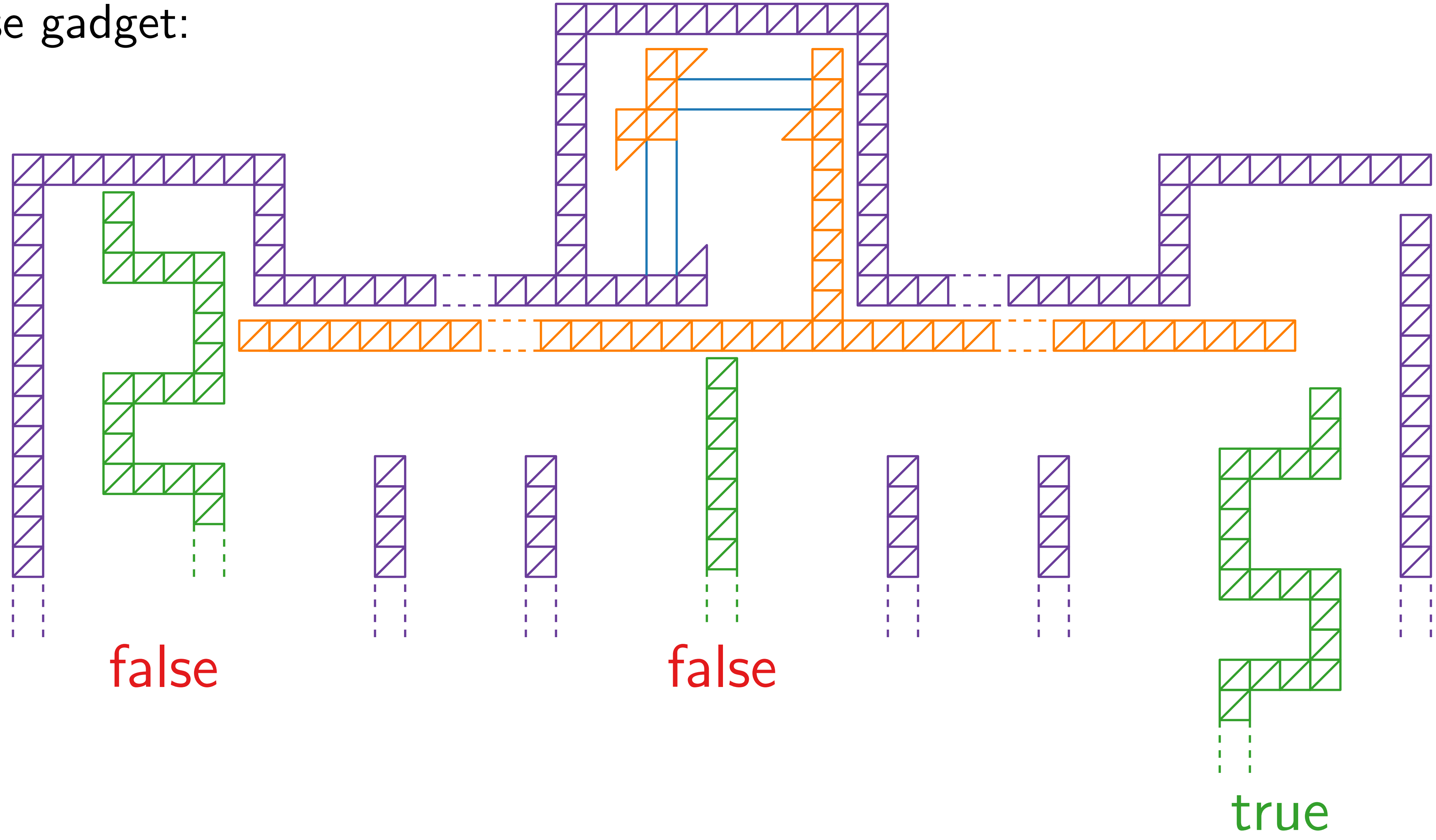
# Outerplanar digraphs

Clause gadget:



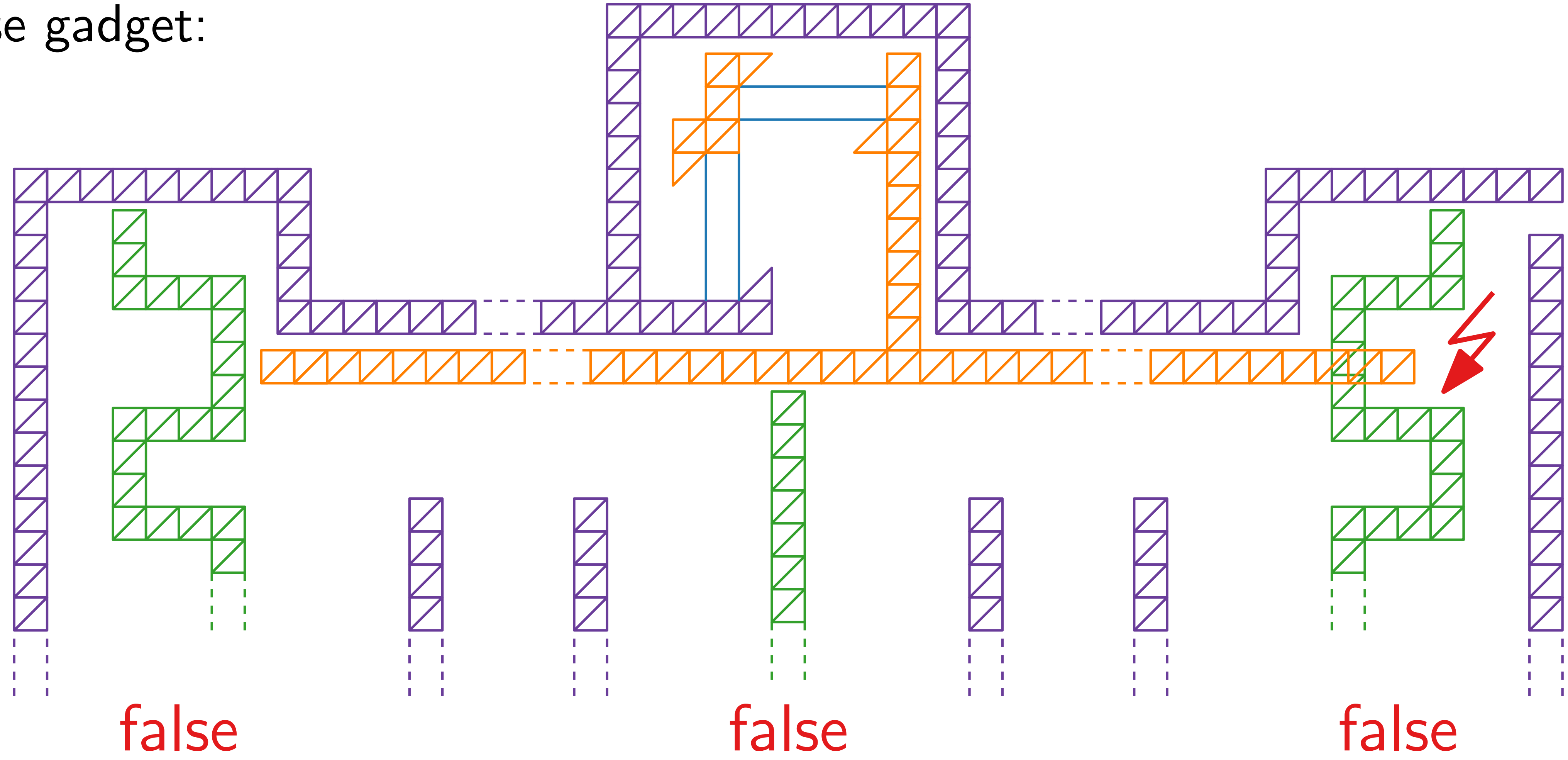
# Outerplanar digraphs

Clause gadget:

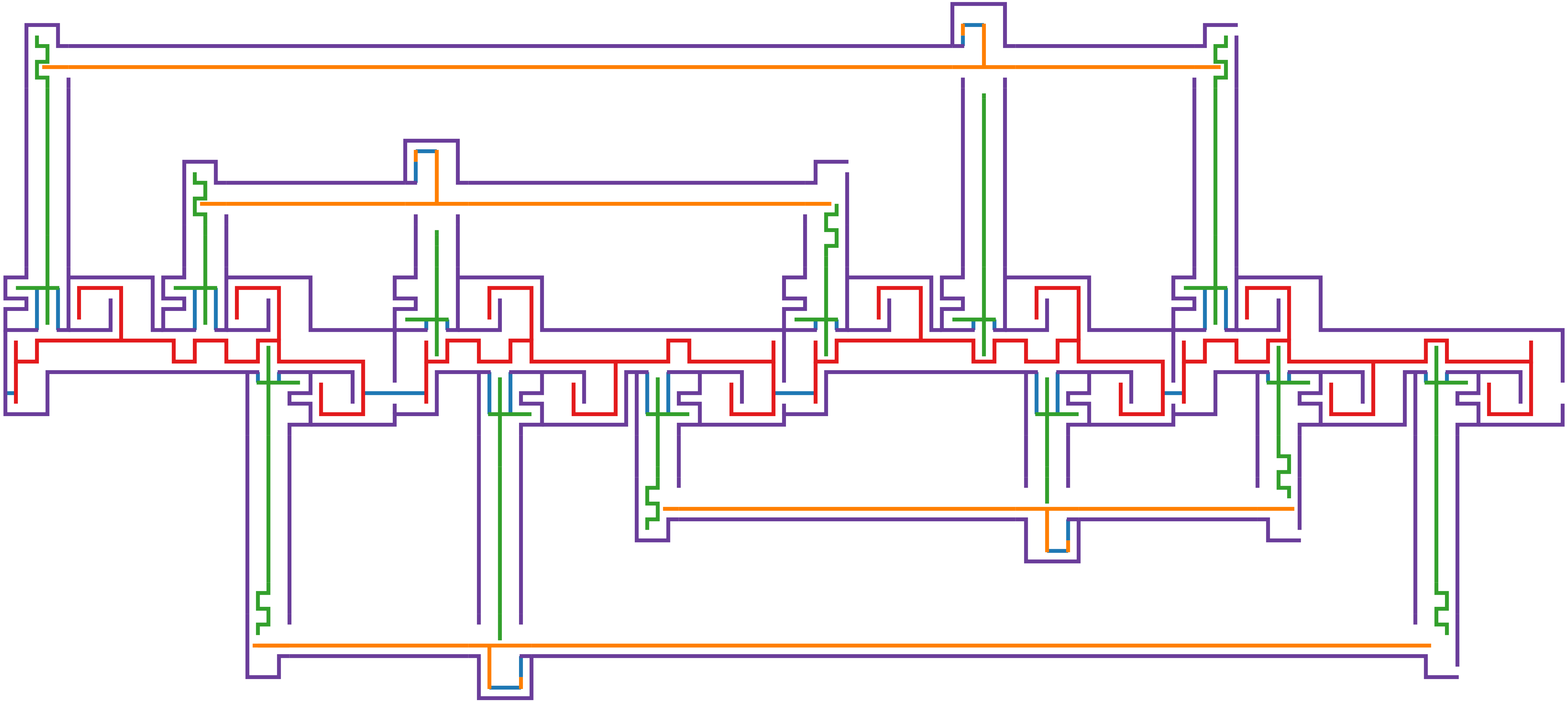


# Outerplanar digraphs

Clause gadget:



# Outerplanar digraphs

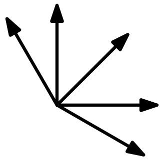


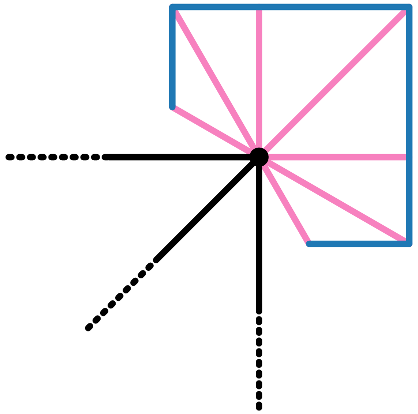
# Planar digraphs

**Theorem.**
Given an upward planar digraph $G$, it is **NP-hard** to decide if $G$ admits an upward planar $k$-slope drawing for $k \geq 3$ if an embedding is specified

# Planar digraphs

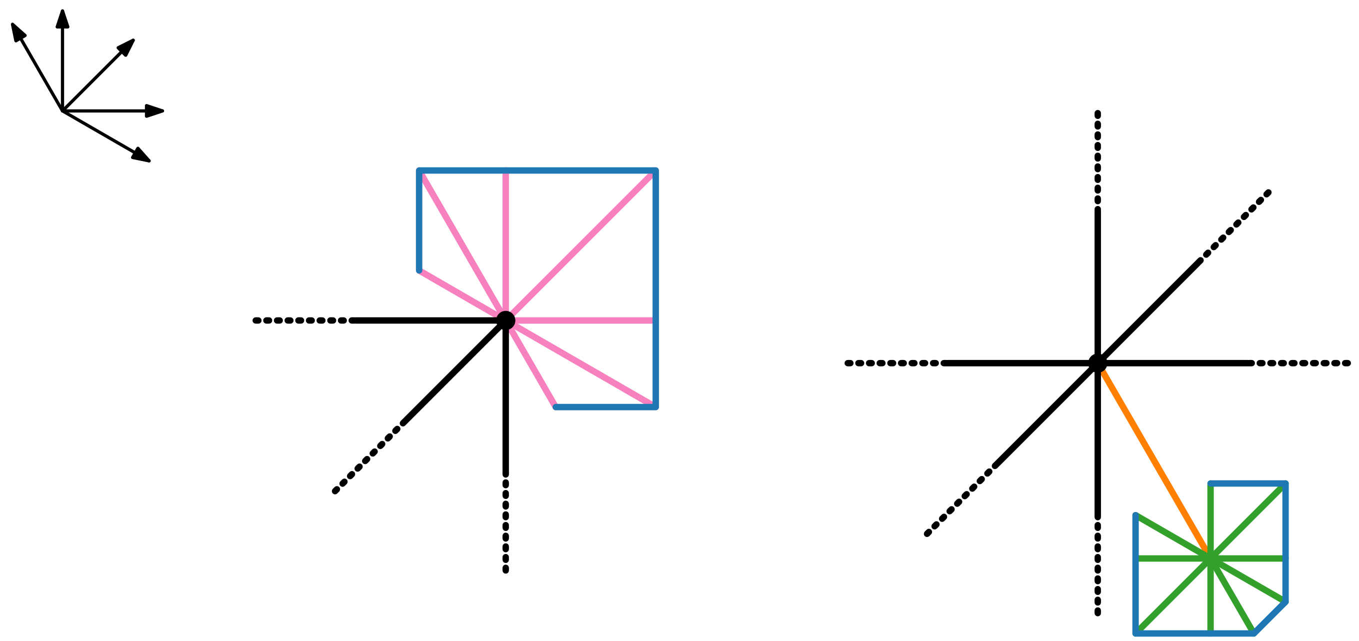
**Theorem.**
Given an upward planar digraph $G$, it is **NP-hard** to decide if $G$ admits an upward planar $k$-slope drawing for $k \geq 3$ if an embedding is specified





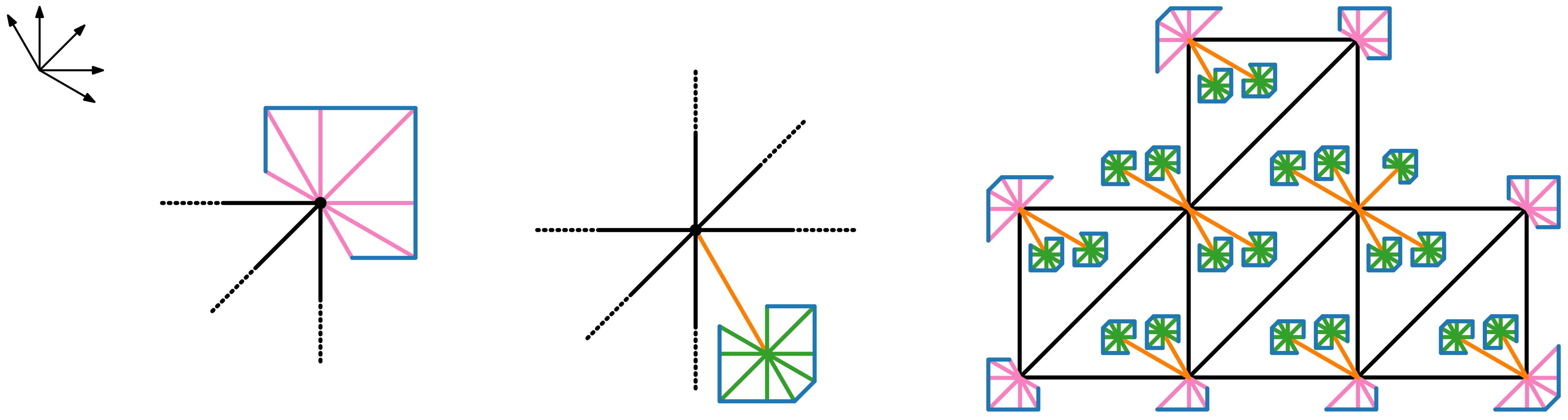
# Planar digraphs

**Theorem.**
Given an upward planar digraph $G$, it is **NP-hard** to decide if $G$ admits an upward planar $k$-slope drawing for $k \geq 3$ if an embedding is specified



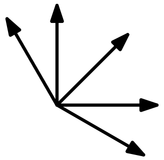
# Planar digraphs

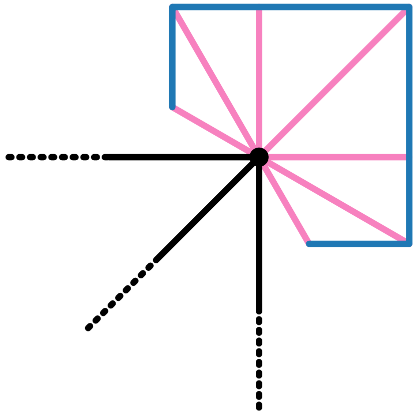
**Theorem.**
Given an upward planar digraph $G$, it is **NP-hard** to decide if $G$ admits an upward planar $k$-slope drawing for $k \geq 3$ if an embedding is specified

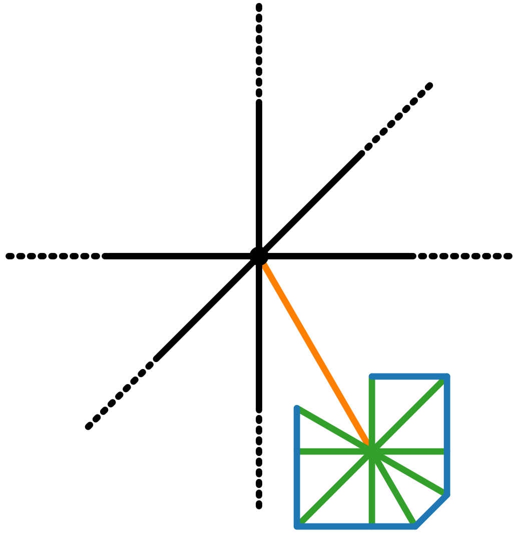


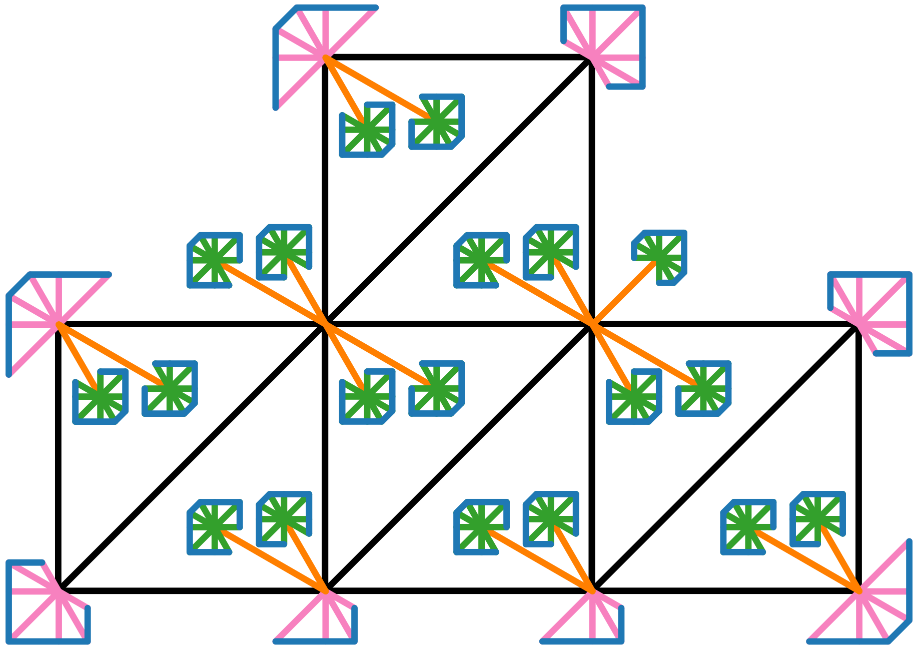
# Planar digraphs

**Theorem.**
Given an upward planar digraph $G$, it is **NP-hard** to decide if $G$ admits an upward planar $k$-slope drawing for $k \geq 3$ if an embedding is specified and for $k \in \mathbb{N} \setminus \{1, 2, 4\}$ otherwise.









# Open questions

- Problem also hard for outerplanar digraphs and $k \geq 4$?
- ... for upward planar digraphs without fixed embedding and $k = 4$?

# Open questions

- Problem also hard for outerplanar digraphs and $k \geq 4$?
- . . . for upward planar digraphs without fixed embedding and $k = 4$?

- Drawings of cactus graph on grid?
- Area requirements of tree drawings?

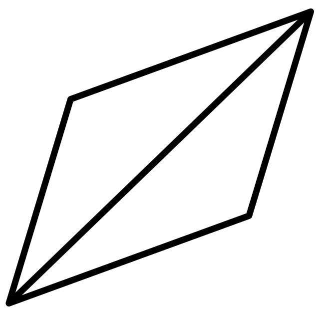
# Open questions

- Problem also hard for outerplanar digraphs and $k \geq 4$?
- ... for upward planar digraphs without fixed embedding and $k = 4$?

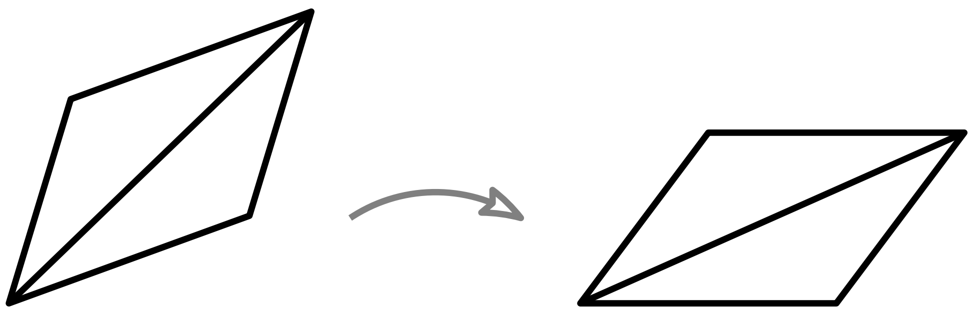
- Drawings of cactus graph on grid?
- Area requirements of tree drawings?

- What about outerpaths?
- What if we drop planarity?
- For fixed $k$, study the segment number
- Allow $\ell$ bends per edge
- ...

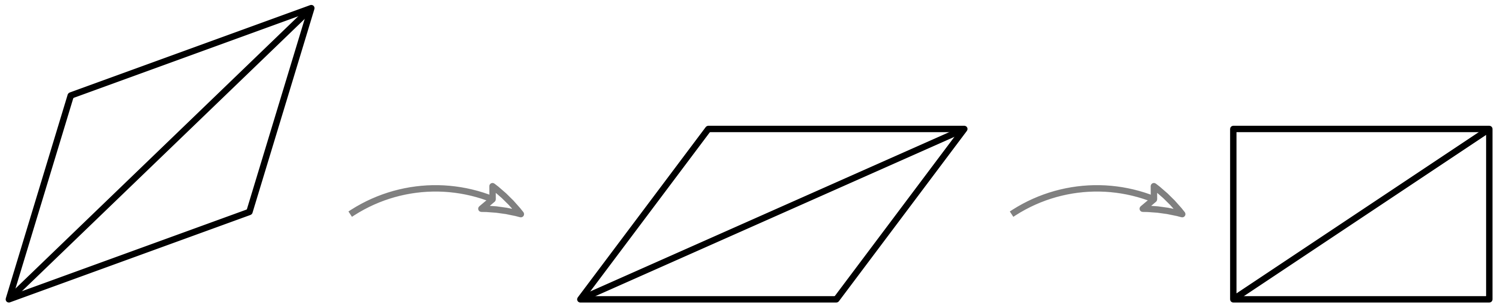
# Slope set



# Slope set



# Slope set



# Slope set